Snowboarding Down A Slope
2010 Skadron Prize In Computational Physics

A snowboard going down a ski slope with moguls can be modeled as a point object with mass m moving on a two-dimensional surface defined by

$$z = h(x, y),$$

where x and y are the coordinates along two horizontal directions that are perpendicular to each other while z is the coordinate along the vertical direction and the function h gives the height of the slope at a point whose horizontal coordinates are x and y. The snowboard experiences the gravitational force due to its mass as well as a frictional and a normal force both exerted by the slope. By numerically solving $\mathbf{F} = m\mathbf{a}$ for the snowboard, we can follow its motion on the slope.

For this year's Skadron prize, we ask you to create a FORTRAN subroutine with which you steer the snowboard after every second by rotating its velocity vector in the tangent plane to the slope at the location of the snowboard by an angle up to $\pm \pi/6$ radians $(= \pm 30°)$. The goal is to complete a snowboard course as fast as you can. For the competition, we choose the course to be a rectangle defined by

$$0 \le x \le 500 \text{ m} \qquad \text{and} \qquad -20 \text{ m} \le y \le 20 \text{ m}.$$

We also choose the function h, which defines the surface of the course, to be

$$h(x, y) = -ax - dy - b\cos(px)\cos(qy),$$

where $a = 0.25$, $d = 0.01$, $b = 0.5 \text{ m}$, $p = \pi/5 \text{ m}^{-1}$, and $q = \pi/2 \text{ m}^{-1}$. The course is mostly sloped along the x-direction with a negative slope of –0.25 and slightly sloped along the y-direction with a negative slope of –0.01. The last term, $-b\cos(px)\cos(qy)$, models moguls, each of which has a height of $2b = 1 \text{ m}$ while their lengths along the downslope or x direction are $\pi/p = 5 \text{ m}$ each and their widths along the cross-slope or y direction are $2\pi/q = 4 \text{ m}$ each.

Rules of the competition
(1) Your snowboard must stay inside the course.
(2) You must complete the course by crossing the finish line at $x = 500 \text{ m}$ within 200 seconds.
(3) The snowboard starts at $(x_0, y_0, z_0) = (0, y_0, z_0)$, where y_0 is randomly selected from an interval $[1.5 \text{ m, } 2.5 \text{ m}]$ and z_0 is given by $z_0 = h(0, y_0)$, and with an initial velocity of

$$(u_0, v_0, w_0) = \left(0, \ 0.5 \text{ m/s}, \ \left(\frac{\partial h}{\partial y}\right)\bigg|_{(x_0, y_0)} (0.5 \text{ m/s})\right).$$

(4) Each contestant will be given the same three sets of initial conditions for the snowboard. The total computation time for the three runs starting with these initial conditions must be within 5 minutes.
(5) The first prize goes to the contestant whose subroutine gets the snowboard to finish the course in the fastest time.

Challenge

Write a FORTRAN subroutine "angle.f" that must start with the following line:

subroutine angle (theta, xa, ya, za, ua, va, wa, xi, yi, ui, vi, it)

where theta, xa, ya, za, ua, va, wa, xi, yi, ui, and vi are all declared as double precision variables in the main program, "skmain10.f," (to be provided by the committee) whereas "it" is declared as an integer variable.

theta is the angle by which you want to rotate the current velocity vector, (ua, va, wa), of the snowboard in the tangent plane to the slope at the current location, (xa, ya, za), of the snowboard.

xi, yi, ui, and vi are array variables of size "it". (xi(j), yi(j)) and (ui(j), vi(j)), where $1 \le j \le it$, provide the recent history of the horizontal component of the position vector and the horizontal component of the velocity vector of the snowboard during one second since the last time you have steered the snowboard. They represent the numerical solution of the equations of motion for (x, y, u, v) and are generated by a 4-th order Runge-Kutta subroutine with a time step size of $dt = 0.01 \sec$ called from the main program.

The Skadron Prize committee will run your subroutine "angle.f" with the main program "skmain10.f" on our computer "meitner." The main program checks if the angle, theta, selected by your subroutine is inside the allowed range of $\left[-\pi/6 \text{ rad}, \pi/6 \text{ rad} \right]$, numerically solves the equations of motion for the snowboard using the 4-th order Runge-Kutta subroutine, and checks if the snowboard remains in the course as well as if the snowboard has crossed the finish line.

To test your subroutine "angle.f" on our computer "meitner":

1. Copy the main program "skmain10.f" on "meitner" to your account on "meitner":

    copy ~hmb/hmb/skmain10.f skmain10.f

2. Compile and run your subroutine with the main program:

    g95 skmain10.f angle.f
    a.out

Prizes: $ 200 for the first place, $100 for the second place. The winners will be announced at the annual physics department award ceremony on Friday, April 30, 2010.

Who can participate: Physics majors at ISU.

Deadline: submit your subroutine by attaching it to an email addressed to

hmb@phy.ilstu.edu

by noon on Tuesday, April 27, 2010.

Appendix 1: Plotting a trajectory of the snowboard using Mathematica or Kaleidagraph

To visualize the trajectory of the snowboard, first copy the data file "snowboard.dat" generated by the main program to an iMac you are using. You can then use "Mathematica" or "Kaleidagraph" to plot the trajectory.

To use "Mathematica," you need to download a "Mathematica" notebook "Skadron10.nb" posted on the following web page:

> www.phy.ilstu.edu/~hmb/hmb.html

By clicking on the icon for the notebook, you can launch "Mathematica."

To make "Mathematica" execute a command or a set of commands (not the header describing the corresponding task), move the cursor to a bracket to the right of the command or commands near the right edge of the page and click on it and then press the "enter" key (not the "return" key) located at the lower right corner of the keyboard.

You can also use "Kaleidagraph" to plot the trajectory as follows:

1. Launch "Kaleidagraph."
2. Use the "Open" command in the "File" pull-down menu in "Kaleidagraph" to open "snowboard.dat." Before locating "snowboard.dat," enable "All Documents" in the dialog box.
3. In the "Text File Input Format" dialog box, select "Space" for "Delimiter:" and ">=1" for "Number:". Type in "1" in the box below "Lines Skipped" and deselect "Read Titles" under "Options:".
4. You find four columns filled with numbers: the column 1 lists the elapsed time in seconds for the snowboard run; the column 2 lists the x-coordinate of the snowboard; the column 3 lists the y-coordinate of snowboard; the column 4 lists the x-component of the velocity vector of the snowboard; the column 5 lists the y-component of the velocity vector of the snowboard. You can plot the trajectory using a linear line plot for the columns 2 and 3.

Appendix 2: A model for the snowboard by Edward Lorenz

A model of the snowboard presented below is described in detail in Chapter 2 of "The Essence of Chaos" (the University of Washington Press, 1993) by Edward Lorenz, who, in 1972, gave a talk, "Predictability: Does the Flap of a Butterfly's Wings in Brazil Set off a Tornado in Texas?", out of which the term "the butterfly effect" came into being.

The equations of motion for the snowboard are

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$

and

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{a} = \mathbf{F}(\mathbf{r}, \mathbf{v}),$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{v} = (u, v, w)$ are the position and the velocity vectors of the snowboard and the net force $\mathbf{F}$ exerted on the snowboard is

$$\mathbf{F}(\mathbf{r},\mathbf{v}) = -mg\hat{z} + m\mathbf{f} - mc\mathbf{v},$$

where $-mg\hat{z}$ is the weight or gravitational force on the snowboard, $-mc\mathbf{v}$ is a frictional force on the snowboard by the slope, and $m\mathbf{f}$ is the normal force on the snowboard by the slope. For the Skadron prize contest, we choose the value for c to be $c = 0.5 \ \mathrm{sec}^{-1}$.

By assuming that the snowboard always remains on the slope, we can reduce the equations of motion to the following 4 equations (for a detailed derivation, see below):

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{du}{dt} = -f\left(\frac{\partial h}{\partial x}\right) - cu, \qquad \frac{dv}{dt} = -f\left(\frac{\partial h}{\partial y}\right) - cv,$$

where

$$f = \frac{g + \left\{\left(\frac{\partial^2 h}{\partial x^2}\right)u^2 + 2\left(\frac{\partial^2 h}{\partial x \partial y}\right)uv + \left(\frac{\partial^2 h}{\partial y^2}\right)v^2\right\}}{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}.$$

We can find z and w by

$$z = h(x,y) \qquad \text{and} \qquad w = \left(\frac{\partial h}{\partial x}\right)u + \left(\frac{\partial h}{\partial y}\right)v.$$

A detailed derivation of the reduced equations of motion for the snowboard

The second equation of motion mentioned above can be rewritten as

$$\frac{d\mathbf{v}}{dt} = -g\hat{z} + \mathbf{f} - c\mathbf{v}.$$

We can express $\mathbf{f}$ as

$$\mathbf{f} = f\left\{-\left(\frac{\partial h}{\partial x}\right)\hat{x} - \left(\frac{\partial h}{\partial y}\right)\hat{y} + \hat{z}\right\}$$

by noting that $\mathbf{f}$ is perpendicular to the following tangent vectors to the slope:

$$\mathbf{X} = \hat{x} + \left(\frac{\partial h}{\partial x}\right)\hat{z} \qquad \text{and} \qquad \mathbf{Y} = \hat{y} + \left(\frac{\partial h}{\partial y}\right)\hat{z},$$

so that

$$\mathbf{f} = f(\mathbf{X} \times \mathbf{Y}) = f\left[\left\{\hat{x} + \left(\frac{\partial h}{\partial x}\right)\hat{z}\right\} \times \left\{\hat{y} + \left(\frac{\partial h}{\partial y}\right)\hat{z}\right\}\right]$$

$$= f\left\{-\left(\frac{\partial h}{\partial x}\right)\hat{x} - \left(\frac{\partial h}{\partial y}\right)\hat{y} + \hat{z}\right\}$$

The equations of motion then become

$$\frac{dx}{dt} = u, \qquad\qquad \frac{dy}{dt} = v, \qquad\qquad \frac{dz}{dt} = w$$

$$\frac{du}{dt} = -f\left(\frac{\partial h}{\partial x}\right) - cu, \qquad \frac{dv}{dt} = -f\left(\frac{\partial h}{\partial y}\right) - cv, \qquad \frac{dw}{dt} = -g + f - cw$$

Assuming that the snowboard always remains on the slope so that $z = h(x,y)$ we find w as a function of u and v:

$$w = \frac{dz}{dt} = \frac{dh}{dt} = \left(\frac{\partial h}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial h}{\partial y}\right)\frac{dy}{dt} = \left(\frac{\partial h}{\partial x}\right)u + \left(\frac{\partial h}{\partial y}\right)v.$$

Note that the velocity vector, $\mathbf{v} = (u, v, w)$ is always perpendicular to $\mathbf{X} \times \mathbf{Y}$:

$$(\mathbf{X} \times \mathbf{Y})\cdot \mathbf{v} = \left\{-\left(\frac{\partial h}{\partial x}\right)\hat{x} - \left(\frac{\partial h}{\partial y}\right)\hat{y} + \hat{z}\right\}\cdot \left\{u\hat{x} + v\hat{y} + w\hat{z}\right\} = 0,$$

so that the velocity vector is on the tangent plane to the slope.

We can express f in terms of u, v, and partial derivatives of h as follows. First, by taking derivatives of the both sides of the above equation for w with respect to t, we find

$$\begin{aligned}
\frac{dw}{dt} &= \left(\frac{\partial h}{\partial x}\right)\frac{du}{dt} + \left(\frac{\partial h}{\partial y}\right)\frac{dv}{dt} + \left\{\left(\frac{\partial^2 h}{\partial x^2}\right)\frac{dx}{dt} + \left(\frac{\partial^2 h}{\partial x\,\partial y}\right)\frac{dy}{dt}\right\}u + \left\{\left(\frac{\partial^2 h}{\partial x\,\partial y}\right)\frac{dx}{dt} + \left(\frac{\partial^2 h}{\partial y^2}\right)\frac{dy}{dt}\right\}v \\
&= \left(\frac{\partial h}{\partial x}\right)\left\{-f\left(\frac{\partial h}{\partial x}\right) - cu\right\} + \left(\frac{\partial h}{\partial y}\right)\left\{-f\left(\frac{\partial h}{\partial y}\right) - cv\right\} + \left\{\left(\frac{\partial^2 h}{\partial x^2}\right)u^2 + 2\left(\frac{\partial^2 h}{\partial x\,\partial y}\right)uv + \left(\frac{\partial^2 h}{\partial y^2}\right)v^2\right\} \\
&= -f\left\{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right\} - cw + \left\{\left(\frac{\partial^2 h}{\partial x^2}\right)u^2 + 2\left(\frac{\partial^2 h}{\partial x\,\partial y}\right)uv + \left(\frac{\partial^2 h}{\partial y^2}\right)v^2\right\}
\end{aligned}$$

Substituting this expression for dw/dt into the following equation of motion,

$$\frac{dw}{dt} = -g + f - cw,$$

we find

$$-g + f - cw = -f\left\{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2\right\} - cw + \left\{\left(\frac{\partial^2 h}{\partial x^2}\right)u^2 + 2\left(\frac{\partial^2 h}{\partial x\,\partial y}\right)uv + \left(\frac{\partial^2 h}{\partial y^2}\right)v^2\right\}.$$

Solving this equation for f, we then obtain

$$f = \frac{g + \left\{\left(\frac{\partial^2 h}{\partial x^2}\right)u^2 + 2\left(\frac{\partial^2 h}{\partial x \partial y}\right)uv + \left(\frac{\partial^2 h}{\partial y^2}\right)v^2\right\}}{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}.$$

The equations of motion finally become

$$\frac{dx}{dt} = u \qquad\qquad \frac{dy}{dt} = v$$

$$\frac{du}{dt} = -f\left(\frac{\partial h}{\partial x}\right) - cu \qquad\qquad \frac{dv}{dt} = -f\left(\frac{\partial h}{\partial y}\right) - cv$$

We can also find z and w by

$$z = h(x, y) \qquad \text{and} \qquad w = \left(\frac{\partial h}{\partial x}\right)u + \left(\frac{\partial h}{\partial y}\right)v.$$

For the Skadron prize contest:

$$h(x, y) = -ax - dy - b\cos(px)\cos(qy),$$

so that

$$\frac{\partial h}{\partial x} = -a + pb\sin(px)\cos(qy), \qquad\qquad \frac{\partial h}{\partial y} = -d + qb\cos(px)\sin(qy)$$

$$\frac{\partial^2 h}{\partial x^2} = p^2 b\cos(px)\cos(qy), \qquad\qquad \frac{\partial^2 h}{\partial y^2} = q^2 b\cos(px)\cos(qy)$$

$$\frac{\partial^2 h}{\partial x \partial y} = -pqb\sin(px)\sin(qy).$$

Appendix 3: How to rotate the velocity vector in a tangent plane

For those who are curious about how the main program rotates the velocity vector by an angle θ, which is to be supplied by a contestant's subroutine, on a tangent plane to the slope at the location of the snowboard, here is the equation for the new velocity vector the main program uses.

$$\mathbf{v}' = u'\hat{x} + v'\hat{y} + w'\hat{z},$$

where

$$u' = (\cos\theta)u - (\sin\theta)\frac{v + \left(\frac{\partial h}{\partial y}\right)w}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}}$$

$$v' = (\cos\theta)v + (\sin\theta)\frac{u + \left(\dfrac{\partial h}{\partial x}\right)w}{\sqrt{\left(\dfrac{\partial h}{\partial x}\right)^2 + \left(\dfrac{\partial h}{\partial y}\right)^2 + 1}}$$

and

$$w' = \left(\frac{\partial h}{\partial x}\right)u' + \left(\frac{\partial h}{\partial y}\right)v'.$$

Derivation

$$\mathbf{v}' = u'\hat{x} + v'\hat{y} + w'\hat{z} = \left(|\mathbf{v}|\cos\theta\right)\frac{\mathbf{v}}{|\mathbf{v}|} + \left(|\mathbf{v}|\sin\theta\right)\left(\hat{n} \times \frac{\mathbf{v}}{|\mathbf{v}|}\right)$$

$$= (\cos\theta)\mathbf{v} + (\sin\theta)\left[\frac{-\left(\dfrac{\partial h}{\partial x}\right)\hat{x} - \left(\dfrac{\partial h}{\partial y}\right)\hat{y} + \hat{z}}{\sqrt{\left(\dfrac{\partial h}{\partial x}\right)^2 + \left(\dfrac{\partial h}{\partial y}\right)^2 + 1}} \times \left(u\hat{x} + v\hat{y} + w\hat{z}\right)\right]$$

$$= (\cos\theta)\mathbf{v}$$

$$+ (\sin\theta)\frac{-\left[v + \left(\dfrac{\partial h}{\partial y}\right)w\right]\hat{x} + \left[u + \left(\dfrac{\partial h}{\partial x}\right)w\right]\hat{y} + \left[-\left(\dfrac{\partial h}{\partial x}\right)v + \left(\dfrac{\partial h}{\partial y}\right)u\right]\hat{z}}{\sqrt{\left(\dfrac{\partial h}{\partial x}\right)^2 + \left(\dfrac{\partial h}{\partial y}\right)^2 + 1}}$$

where $\hat{n}$ is a unit vector normal to the slope and is defined by

$$\hat{n} \equiv \frac{\mathbf{X} \times \mathbf{Y}}{|\mathbf{X} \times \mathbf{Y}|}.$$