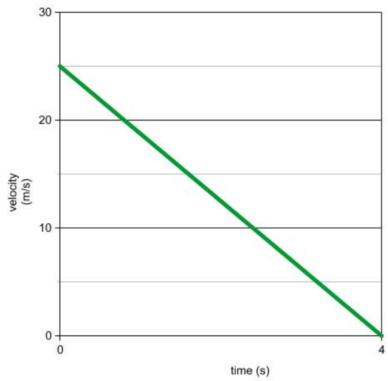
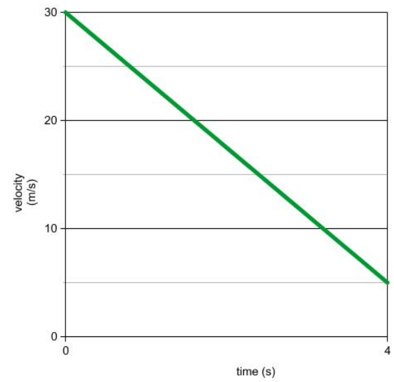
Finding the Displacement of a Uniformly Accelerated Object

(a = constant, including 0)



Graph 1



Graph 2

An object moves with a uniformly accelerated motion and has a $v(t) = (-6.25m/s)\Delta t + 25m/s$ as shown in **Graph 1**. The area under the line in this object's velocity-time graph is equal to its displacement (Δx). There are two simple methods (not involving calculus) that we can use to arrive at the value of the displacement:

1) Use kinematics:

From kinematics we know that $\Delta x = \bar{v}\Delta t$ which, for a uniformly accelerating body, is the same as $\Delta x = \frac{1}{2}(v_i + v_f)\Delta t$.

From **Graph 1** we have $\Delta x = \frac{1}{2}\left(25m/s + 0m/s\right)(4s - 0s) = \frac{1}{2}(25m/s)(4s) = 50m$

2) Use geometry and the graph:

From the formula for area of a triangle we know that $\Delta x = \frac{1}{2}hb = \frac{1}{2}\left(25m/s - 0m/s\right)(4s - 0s) = \frac{1}{2}(25m/s)(4s) = 50m$

While the result is the same in each of the above approaches, **how does one reconcile the obvious difference in the + and – signs on the 0m/s terms?** *This is a special case due to the fact that the base of the triangle is the X-axis.* Consider the more general situation where the base of the displacement triangle is not the X-axis. This can be seen in **Graph 2** where the Y-intercept is 30m/s and the velocity is 5m/s at 4s. The slope is the same; only the Y-intercept is greater by 5m/s.

Now, the area under the line in **Graph 2** is the sum of two areas, the area of the triangle plus the area of the rectangle below it. Consider again the geometric solution:

$$\text{displacement} = \text{area under triangle} + \text{area of rectangle}$$

$$\Delta x = \frac{1}{2}h_t b + h_r b$$

$$\Delta x = \frac{1}{2}\left(v_i - v_f\right)\Delta t + v_f\Delta t \qquad (3)$$

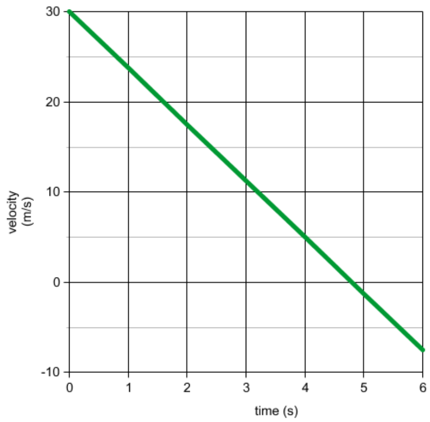
$$\Delta x = \frac{1}{2}v_i\Delta t - \frac{1}{2}v_f\Delta t + v_f\Delta t$$

$$\Delta x = \frac{1}{2}v_i\Delta t + \frac{1}{2}v_f\Delta t$$

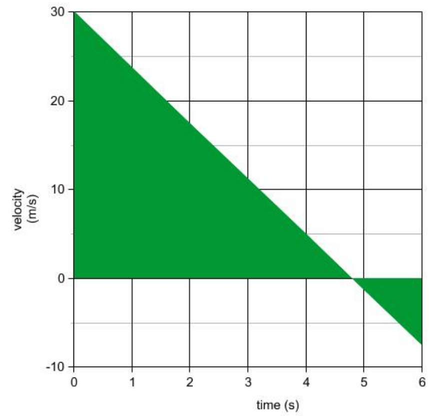
$$\Delta x = \frac{1}{2}(v_i + v_f)\Delta t$$

When there is no rectangle under the triangle (e.g., $v_f = 0$), the solution in line 3 above reduces to $\Delta x = \frac{1}{2}(v_i - v_f)\Delta t$. So, the solutions – those with the +0m/s and the –0m/s in methods 1 and 2 above – are both legitimate. The form of $\Delta x = \frac{1}{2}(v_i + v_f)\Delta t$ is the more general, and includes the rectangular area under the triangle if any.

This approach is fine of the "motion" is restricted to the first Cartesian quadrant, but what happens if the motion "enters" the fourth Cartesian quadrant as shown in **Graph 3** and we have displacement area *below* the X-axis?



Graph 3



Graph 4

The equation of motion is the same as in **Graph 2**, $v(t) = \left(-6.25\frac{m}{s^2}\right)\Delta t + 30\frac{m}{s}$, but we have extended the time interval to 6s as shown in **Graph 3**. The displacements above and below the X-axis are shown in **Graph 4**.

What is the displacement between, say, 0s and 6s? Notice that v = 0m/s when t = 4.8s. Note too that the displacement following 4.8s is negative (the object is moving with a negative velocity) while the area of a triangle is always positive. Again, there are two methods to calculate the displacement over the entire interval.

1) Use kinematics:

From kinematics we know that $\Delta x = \bar{v}t$ which, for a uniformly accelerating body, is the same as $\Delta x = \frac{1}{2}(v_i + v_f)\Delta t$.

From **Graph 3** we have $\Delta x = \frac{1}{2}\left(30m/s - 7.5m/s\right)(6s - 0s) = 67.5m$

2) Use geometry and the graph to sum the displacements represented by the two triangles shown in **Graph 4**:

From the formula for area of a triangle (whose sides are considered positive and whose displacements can be either positive or negative) we know that displacement in this case is the area of triangle 1 – area of triangle 2 (because the object is moving with a negative velocity after 4.8s).

$$\Delta x = \frac{1}{2}h_1 b_1 - \frac{1}{2}h_2 b_2$$
$$\Delta x = \frac{1}{2}\left(v_i - v_f\right)_1(\Delta t) - \frac{1}{2}\left(v_i - v_f\right)_2(\Delta t)$$
$$\Delta x = \frac{1}{2}\left(30\frac{m}{s} - 0\frac{m}{s}\right)_1(4.8s - 0s) - \frac{1}{2}\left(0\frac{m}{s} - -7.5\frac{m}{s}\right)_2(6s - 4.8s)$$
$$\Delta x = \frac{1}{2}\left(30\frac{m}{s}\right)(4.8s) - \frac{1}{2}\left(7.5\frac{m}{s}\right)(1.2s)$$
$$\Delta x = 72m - 4.5m$$
$$\Delta x = 67.5m$$

The conclusion that we can draw from this analysis is that $\Delta x = \frac{1}{2}(v_i + v_f)\Delta t$ is a *less confusing, more convenient, and less-error-prone* way of calculating displacement for accelerated motion even if initial or final velocities are less than zero.