

Notes on Calculus (PHY110, Dr. Su)

Summary: Definition: For function $y(x)$:

$$\frac{\Delta y}{\Delta x} = \frac{y_f - y_i}{x_f - x_i} \rightarrow (\text{when } \Delta x \rightarrow 0) \rightarrow \frac{dy}{dx},$$

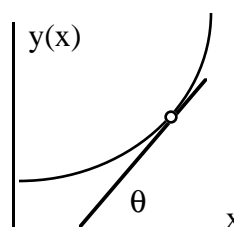
where $y_f \equiv y(x_f)$ and $y_i \equiv y(x_i)$.

Other symbols: $\frac{d}{dx}[y(x)]$, $y'(x)$, $D_x y$ or \dot{y} .

Physical example: $\frac{dx}{dt} = v$ (instantaneous) velocity

signs: + if upward-pointing, - if downward-pointing

Graphical representation: slope of line tangent to curve slope = $\tan \theta = dy / dx$



Some Rules for Differentiation

(below c , n represent any constants, u and v are functions of variable x)

Constant Rule: $\frac{dc}{dx} = 0$

Homogeneous Rule: $\frac{d}{dx}(cu) = c \frac{du}{dx}$

Power Rule: $\frac{d}{dx}(x^n) = n x^{n-1}$

ex1: $\frac{dx}{dx} = (1) x^{1-1} = \frac{x}{x} = 1$ (Identify Rule) ex2: $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1) x^{-2} = -\frac{1}{x^2}$

ex3: $\frac{d\sqrt{x}}{dx} = \frac{d}{dx}(x^{1/2}) = \left(\frac{1}{2}\right) x^{-1/2} = \frac{1}{2\sqrt{x}}$

Sum Rule: $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$ ex: $\frac{d}{dx}(3x^2 + 2) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(2) = 3 \frac{d}{dx}(x^2) + 0 = (3)(2x) = 6x$

Multiplication Rule (also Product Rule or Leibniz Rule): $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

ex: $\frac{d}{dx}(x^2) = \frac{d}{dx}(xx) = x \frac{dx}{dx} + x \frac{dx}{dx} = (x)(1) + (x)(1) = 2x$ (also from Power Rule)

Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \left(v \frac{du}{dx} - u \frac{dv}{dx}\right) / v^2$ ex: $\frac{d}{dx}\left(\frac{1}{v}\right) = \left(v \frac{d1}{dx} - 1 \frac{dv}{dx}\right) / v^2 = -\frac{1}{v^2} \frac{dv}{dx}$ (Reciprocal Rule)

Chain Rule: $\frac{d}{dx}\{y[u(x)]\} = \frac{dy(u)}{du} \frac{du(x)}{dx}$

Derivative of Fundamental Functions

$$\frac{d}{dx}(e^x) = e^x \quad \frac{d}{dx}[\ln(x)] = \frac{1}{x} \quad \frac{d}{dx}[\sin(x)] = \cos(x) \quad \frac{d}{dx}[\cos(x)] = -\sin(x)$$

$$\text{ex: } \frac{d}{dx}[2 \sin(3x + 4)] = 2 \frac{d}{dx}[\sin(3x + 4)] = 2 \frac{d}{du}[\sin(u)] \frac{d}{dx}(3x + 4) = 2 \cos(u)(3) = 6 \cos(3x + 4)$$

display special derivatives of special functions such as, via fortran

sint e^t logt

distance as an integral

Q: Knowing velocity of an object moving along a line, v(t), by closely monitoring the speedometer, can we predict position: x(t) with out using a video tape?

THE MEANING OF A (DEFINITE) INTEGRAL

- (1) Displacement for a constant velocity, v(t) = v, over an interval t₀ to t_f:

$$\text{total displacement} = v \Delta t = v(t_f - t_0)$$

- (2) Displacement for two piecewise constant velocities over Δt₁, and Δt₂:

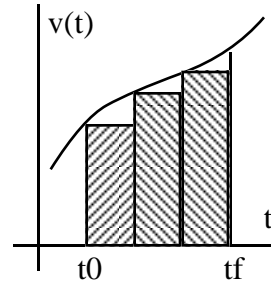
$$\text{total displacement} = v_1 \Delta t_1 + v_2 \Delta t_2 \equiv \sum_{i=1}^2 v_i \Delta t_i \quad (\text{notation for summation})$$

- (3) Displacement for a continuous varying velocity v(t) over an interval t₀ to t_f, divided up into a large number of N-intervals. Within each interval velocity is approximately constant.

$$\begin{aligned} \text{total displacement} &= v(t_1) \Delta t_1 + v(t_2) \Delta t_2 + \dots + v(t_N) \Delta t_N \\ &= \sum_{i=1}^N v(t_i) \Delta t_i \xrightarrow{\Delta t_i \rightarrow 0 \text{ and } N \rightarrow \infty} \int_{t_0}^{t_f} v(t) dt \end{aligned}$$

The last limit is called a (definite) **integral** of v(t) over t, latin summa. Note that again “d” in dt means that we can make time interval as small as we want, and we add up all the small distances to get the.

Graphical interpretation: area under the curve



$v(t)$ b/n lines $t = t_0$ and $t = t_f$.

HOW TO EVALUATE AN INTEGRAL

example: 1-d displacement for a time varying velocity $v(t)$

$$\text{total displacement} = \sum_{i=1}^N v(t_i) \Delta t_i \xrightarrow{\Delta t_i \rightarrow 0 \text{ and } N \rightarrow \infty} \int_{t_0}^{t_f} v(t) dt$$

(1) Sum for 3 intervals separated by coordinates x_0, x_1, x_2, x_f :

$$\begin{aligned} \text{total displacement} &= \sum_{i=1}^3 v(t_i) \Delta t_i = \sum_{i=1}^3 \Delta x_i = (x_1 - x_0) \\ &\quad + (x_2 - x_1) \\ &\quad + (x_f - x_2) \\ &= x_f - x_0 = x(t_f) - x(t_0) \equiv [x(t)]_{t_0}^{t_f} \equiv x(t) \Big|_{t_0}^{t_f} \end{aligned}$$

(2) For time varying velocity $v(t)$:

$$\int_{t_0}^{t_f} v(t) dt = [x(t)]_{t_0}^{t_f}, \text{ where } v(t) = \frac{dx(t)}{dt}$$

Relationship between the **integrand** and the (indefinite) **integral**:

$$\begin{array}{ccc} & \xrightarrow{\text{the derivative of}} & \\ v(t) & & x(t) \\ & \xleftarrow{\text{the anti-derivative of}} & \end{array}$$

Note: **d** seems to **undo** \int .

Note: Every function may be differentiated analytically. But not every integral may be integrated in terms of simple functions already been given names.

Note: It is always possible to numerically find the sum to any accuracy.

BASIC RULES OF (DEFINITE) INTEGRAL

$$\int_a^b f'(x) dx = [f(x)]_a^b \quad (\text{note: the LHS} = \int_a^b \frac{d[f(x)]}{dx} dx = \int_a^b d[f(x)])$$

Homogeneous Rule: $\int_a^b c f(x) dx = c \int_a^b f(x) dx \quad \int 2x dx = 2 \int x dx$

Additive Rule: $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx \quad \int (x^2 + x^3) dx = \int x^2 dx + \int x^3 dx$

Linear Rule: $\int_a^b [c_1 f(x) + c_2 g(x)] dx = c_1 \int_a^b f(x) dx + c_2 \int_a^b g(x) dx \quad \int (5x^2 + 6x^3) dx = 5 \int x^2 dx + 6 \int x^3 dx$

Additivity with respect to the interval of integration: $\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx \quad \int_0^2 = \int_0^1 + \int_1^2$

Power Rule: $\int_a^b x^n dx = \left[\frac{x^{n+1}}{n+1} \right]_a^b$ (for $n \neq -1$) $\int x dx = x^2/2$ ex: $n=0: \int_a^b dx = [x]_a^b$

special for $n=-1$: $\int_a^b \frac{dx}{x} = [\ln x]_a^b = \ln b - \ln a = \ln\left(\frac{b}{a}\right)$

Special integrals: **Fortran** demo:

$$\int_a^b e^x dx = [e^x]_a^b \quad \int_a^b \cos(x) dx = [\sin(x)]_a^b \quad \int_a^b \sin(x) dx = [-\cos(x)]_a^b$$

APPLICATIONS of INTEGRATIONS

(check which integration rules have been applied)

ex: velocity of a particle under the constant acceleration, a:

$$\frac{dv}{dt} = a, \text{ so } dv = a dt \quad \text{integrate from time 0 to time t: } \int_0^t dv = \int_0^t a dt$$

$$[v(t)]_0^t = a \int_0^t dt = a [t]_0^t \quad \text{we get} \quad v(t) - v_0 = a t$$

or $v(t) = v_0 + a t$ (kinematic equation)

ex: displacement of the previous system:

$$\frac{dx}{dt} = v(t) = v_0 + a t, \quad \text{so } dx = (v_0 + a t) dt, \quad \text{integrate from time 0 to time t:}$$

$$\int_0^t dx = \int_0^t (v_0 + a t) dt = \int_0^t (v_0) dt + \int_0^t (a t) dt = v_0 \int_0^t dt + a \int_0^t t dt = v_0 [t]_0^t + a \left[\frac{t^2}{2} \right]_0^t = v_0 t + \frac{1}{2} a t^2 =$$

and the LHS = $[x(t)]_0^t = x(t) - x_0$

or $x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$ (kinematic equation)