INTEGRAL CALCULUS FOR PHY110 BY DR. SU

Q: Knowing velocity of an object moving along a line, v(t), can we predict displacement: x(t)-x(0)?

THE MEANING OF A (DEFINITE) INTEGRAL

(1) Displacement for a constant velocity, v(t) = v, over an interval t_0 and t_f:

$$\text{total displacement} = v\,\Delta t = v\left(t_f - t_0\right)$$

(2) Displacement for two piecewise constant velocities over Δt_1, and Δt_2:

$$\text{total displacement} = v_1\,\Delta t_1 + v_2\,\Delta t_2 \equiv \sum_{i=1}^{2} v_i\,\Delta t_i \quad \text{(notation for summation)}$$

(3) Displacement for a continuous varying velocity v(t) over an interval t_0 and t_f, divided up into a large number of N-intervals. Within each interval velocity is approximately constant.

$$\begin{aligned}\text{total displacement} &= v(t_1)\,\Delta t_1 + v(t_2)\,\Delta t_2 + \cdots + v(t_N)\,\Delta t_N \\ &= \sum_{i=1}^{N} v(t_i)\,\Delta t_i \xrightarrow{\Delta t_i \to 0 \text{ and } N \to \infty} \int_{t_0}^{t_f} v(t)\,dt\end{aligned}$$

The last limit is called a (definite) **integral** of v(t) over t.

Graphical interpretation: area under the curve v(t) between lines t = t_0 and t = t_f.

HOW TO EVALUATE AN INTEGRAL

example: 1-d displacement for a time varying velocity v(t)

$$\text{total displacement} = \sum_{i=1}^{N} v(t_i)\,\Delta t_i \xrightarrow{\Delta t_i \to 0 \text{ and } N \to \infty} \int_{t_0}^{t_f} v(t)\,dt$$

(1) Sum for 3 intervals separated by coordinates $x_0,\ x_1,\ x_2,\ x_f$:

$$\begin{aligned}\text{total displacement} &= \sum_{i=1}^{3} v(t_i)\,\Delta t_i = \sum_{i=1}^{3} \Delta x_i = \left(x_1 - x_0\right) \\ &\qquad + \left(x_2 - x_1\right) \\ &\qquad + \left(x_f - x_2\right) \\ &= x_f - x_0 = x(t_f) - x(t_0) \equiv \left[\ x(t)\ \right]_{t_0}^{t_f} \equiv x(t)\Big|_{t_0}^{t_f}\end{aligned}$$

(2) For time varying velocity v(t):

$$\int_{t_0}^{t_f} v(t)\,dt = \left[\ x(t)\ \right]_{t_0}^{t_f}, \text{ where } v(t) = \frac{dx(t)}{dt}$$

Relationship between the **integrand** and the (indefinite) **integral**:

$$v(t)\ \underset{\xleftarrow[\text{the anti-derivative of}]{}}{\xrightarrow{\text{the derivative of}}}\ x(t)$$

BASIC RULES OF (DEFINITE) INTEGRAL

$\int_a^b f'(x)\,dx = [\,f(x)\,]_a^b$ (note: the LHS $= \int_a^b \frac{d[\,f(x)\,]}{dx}\,dx = \int_a^b d[\,f(x)\,]$)

Homogeneous Rule: $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

Additive Rule: $\int_a^b [\,f(x)+g(x)\,]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Linear Rule: $\int_a^b [\,c_1\,f(x)+c_2\,g(x)\,]\,dx = c_1\int_a^b f(x)\,dx + c_2\int_a^b g(x)\,dx$

Additivity with respect to the interval of integration: $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

Power Rule: $\int_a^b x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b$ (for $n \neq 1$) ex: n=0: $\int_a^b dx = [\,x\,]_a^b$

for n=-1: $\int_a^b \frac{dx}{x} = [\,\ln x\,]_a^b = \ln b - \ln a = \ln\left(\frac{b}{a}\right)$

Special integrals:

$\int_a^b e^x\,dx = [\,e^x\,]_a^b$ $\int_a^b \cos(x)\,dx = [\,\sin(x)\,]_a^b$ $\int_a^b \sin(x)\,dx = [\,-\cos(x)\,]_a^b$

APPLICATIONS

(check which integration rules have been applied)

ex: velocity of a particle under the constant acceleration, a:

$\frac{dv}{dt} = a$, so $dv = a\,dt$ integrate from time 0 to time t: $\int_0^t dv = \int_0^t a\,dt$

$[\,v(t)\,]_0^t = a\int_0^t dt = a\,[\,t\,]_0^t$ we get $v(t) - v_0 = a\,t$

or $v(t) = v_0 + a\,t$ (kinematic equation)

ex: displacement of the previous system:

$\frac{dx}{dt} = v(t) = v_0 + a\,t$, so $dx = (v_0 + a\,t)\,dt$, integrate from time 0 to time t:

$$\int_0^t dx = \int_0^t (v_0 + a\,t)\,dt = \int_0^t (v_0)\,dt + \int_0^t (a\,t)\,dt = v_0\int_0^t dt + a\int_0^t t\,dt = v_0\,[\,t\,]_0^t + a\left[\frac{t^2}{2}\right]_0^t = v_0\,t + \frac{1}{2}a\,t^2 =$$

and the LHS $= [\,x(t)\,]_0^t = x(t) - x_0$

or $x(t) = x_0 + v_0\,t + \frac{1}{2}a\,t^2$ (kinematic equation)

ex: work by weight: $\int_0^d mg\,dx = mg\int_0^d dx = mg\,[\,x\,]_0^d = mg\,(d-0) = mgd$

ex: work by elastic force in a spring: $\int_0^d k\,x\,dx = k\int_0^d x\,dx = k\left[\frac{x^2}{2}\right]_0^d = k\left(\frac{d^2}{2} - \frac{0^2}{2}\right) = \frac{1}{2}k\,d^2$