Computer Problem #1
Velocity Dependent Forces and Terminal Velocity

The primary objectives of this numerical experiment are two fold: 1) to get you familiar with a workhorse of numerical analysis (i.e. Runge-Kutta) and 2) to allow you to experiment with drag forces. Attached to this document you will find a fixed step sized fourth order Runge-Routine and its calling program. In its current form, the code is written so that it can solve up to 6 coupled first order differential equations of the form

$$\frac{dQ_n}{ds} = f_n(s, Q_1, Q_2, \ldots)$$

where s is the independent variable and th Q_n are the dependent variables.

Once you have logged on to one of the computers in the department, you need to use the command

ssh meitner.phy.ilstu.edu

to log into meitner where the FORTRAN compiler and codes are located. If you do not have an account on meitner see Dr. Bogue ASAP to set one up. You may also connect to meitner using a terminal emulator such as PuTTy from any computer with an internet connection.

Once on meitner, you may obtain an electronic copy of the program by using the command

cp ~holland/phy220/rk4_90.f rk4_90.f

This will place a copy of the code in your root directory in the filed rk4_90.f. You should keep this file unchanged and make new copies (with different names) for each of the assignments this semester.

In this first assignment we will be working with purely one dimensional motion, albeit with velocity dependent forces, thus our two first order coupled equations of motion are

$$\frac{dv}{dt} = a(x, v; t) \qquad ; \qquad \frac{dx}{dt} = v$$

In terms of our Q_n and s, we may choose $Q_1 = x$, $Q_2 = v$ and $s = t$, yielding

$$\frac{dQ_2}{ds} = a(Q_1, Q_2; s) \qquad ; \qquad \frac{dQ_1}{ds} = Q_2$$

For our investigation, we want to examine the effects of the size of a spherical object on its terminal velocity. For the sake of being specific, let us assume that we are dropping spherical copper balls ($\rho = 8960 kg/m^3$) of varying radii r out of a balloon a height h above the earth. The forces acting on the ball are then gravity, which is pulling downward $F_g = -mg = -\frac{4}{3}\pi r^3 \rho g$ and the drag force, which will be pulling the ball upwards $F_d = -C_1 v - C_2 |v| v$ where $C_1 = 3.1 \times 10^{-4}\ r$ and $C_2 = 0.88\ r^2$. Thus, our equations of motion are

$$\frac{dx}{dt} = v \tag{1}$$

and

$$\frac{dv}{dt} = -g - (C_1/m)v - (C_2/m)|v|v \tag{2}$$

where $m = \frac{4}{3}\pi r^3 \rho$ In class we found that for reasonable size objects (i.e. BB's and bigger) the quadratic term in the drag force dominates the linear term except at very low speeds (e.g. 1 m/s and below). Thus we would expect to find that the terminal velocity should be determined by the quadratic term for macroscopic particles. By balancing the quadratic drag term with gravity one finds that the terminal velocity is given by

$$v_t = 646\sqrt{r}\ m/s \tag{3}$$

where we have substituted in for $\rho = 8960 kg/m^3$ and $g = 9.8\ m/s^2$. If on the other hand, the particles are extremely small (e.g. dust grains and smaller) one might expect that the linear term in the drag force would dominate. Again balancing gravity against drag, we find that the terminal velocity should be

$$v_t = 1.19 \times 10^9 r^2\ m/s \tag{4}$$

There are a few important thing to note in comparing the terminal velocities predicted by equations (3) and (4):

1. The functional dependence of the terminal velocity is vastly different for the two cases. With linear drag, if the radius doubles, the terminal velocity increase by 400% (i.e. a factor of 4), whereas with quadratic drag it would only go up by about 41%.

2. The huge coefficient in front of the r term in the linear drag case indicates that it is only valid for extremely small particles.

3. The terminal velocities predicted by equation (3) and (4) are equal for particles with a radius of approximately $67\mu m$. Hence we would expect that in this general size range, we must account for both linear and quadratic drag to get an accurate determination of the terminal velocity.

Your assignment:

1. Vary the radius of the particle from 0.1 μm to 0.1 m and determine the terminal velocity as a function of the radius. In performing this calculation, keep in mind the following consideration:

   (a) For ease of calculation, we may assume that we are letting the spheres start from rest at $x = 0$. Note that this means if positive x is upward, the particle position has $x \leq 0$. Don't worry about the balls hitting the ground, assume that we are starting high enough that we have plenty of distance to obtain terminal velocity.

(b) Since we are covering six orders of magnitude in scale size, it would behoove you to plot your graphs using log-log axes. Also note that you would be wise to not use evenly space data points in radius but rather to space them equally on a logarithmic scale. As a first pass, I would use 0.1 μm, 0.3 μm, 1.0 μm, 3.0 μm, 10.0 μm,....

(c) Make sure that you allow the code to run long enough for the balls to reach terminal velocity. The easiest way to do this is to put a conditional test in your code which checks to make sure that the particle acceleration is small with respect to gravity. If the acceleration is small, this means you are essentially at terminal velocity. It is interesting to check to see what difference it makes when we change what we mean by "small with respect to gravity".

(d) You will find that as the radii get smaller, you will need to use a smaller time step. Why do you think this is so?

2. Using your small radius results and Kalidagraph make a plot of the terminal velocity as a function of the radius for the entire range of radii.

3. As a homework problem you showed that the terminal velocity of a particle including both linear and quadratic drag is given by

$$v_t = \sqrt{(mg/C_2) + (C_1/2C_2)^2} - (C_1/2C_2)$$

Use the same parameters as given above to calculate the terminal velocity as a function of the radius. (Note: this is most easily done as an additional step in the Runge Kutta routine since you already have to calculate C_1, C_2 and m.)

4. Compare the theoretical and numerical results over the full range of radii. An appropriate method would be to plot the percent difference between the two results.

5. Use only the small radius data (i.e. $r \ll 67\mu m$) to make a log-log plot of the terminal velocity as a function of radius.

6. Use the curve fitting routine in Kalidagraph to fit a power law to the data. Do your results agree with the theoretical arguments presented above?

7. Use only the large radius data (i.e. $r \gg 67\mu m$) to make a log-log plot of the terminal velocity as a function of radius.

8. Use the curve fitting routine in Kalidagraph to fit a power law to the data. Do your results agree with the theoretical arguments presented above?

9. Write up a self contained report describing your results. Assume that the report will be read by an educated lay person who understands the basics of Newton's laws, but has no knowledge of drag forces. At a minimum, the report will have four graphs: 1) terminal velocity as a function of radius for the whole range of radii, 2) some form of a comparison of the theoretical and numerical terminal velocities for the entire range of radii, 3) terminal velocity as a function of the radius for the small radii (this plot should also show the curve fit) and 4) terminal velocity as a function of the radius for the large radii (this plot should also show the curve fit). Include a derivation of equations (3) and (4). How might this information about terminal velocities be applied in more realistic problems. For example what would this tell me about the ash and dust particles that are ejected from a volcano?

   DO NOT just staple together a bunch of computer output an make me interpret it. You will score very poorly if you do that.

   You do not need to include a copy of the code that you are using.