Computer Problem #3
Big Bertha

During WWI, the Germans constructed one of the world's first super weapons; an extremely large cannon that was nicknamed Big Bertha. This remarkable cannon was able to shoot a projectile with a muzzle velocity of 1,450 m/s (approx Mach 4.4). They would typically shoot the cannon at an angle of 55° above the horizontal. A simple calculation (Chapter 4) tells us that the theoretical range for Big Bertha should be

$$R = \frac{v_o^2}{g}\sin(2\theta) = \frac{1450\text{m/s}}{9.8\text{m/s}^2}\sin(110°) = 202\text{km}$$

Additionally, we find that the time of flight for the projectile is

$$T = \frac{2v_0\sin\theta}{g} = \frac{(2)(1450\text{m/s})\sin(55°)}{9.8\text{m/s}^2} = 242\text{s}$$

and the maximum height achieved is

$$\begin{aligned} y(T/2) &= -\frac{g(T/2)^2}{2} + v_0(T/2)\sin\theta \\ &= -\frac{(9.8\text{m/s}^2)(141\text{s})^2}{2} + (1450\text{m/s})(141\text{s})\sin(55°) \\ &= 72\text{km} \end{aligned}$$

This cannon could easily shoot a projectile over Mount Everest! In fact, a height of 72 km would put the projectile into the lower part of the ionosphere. Due to drag forces on the projectile, the actual range of Big Bertha was only 120 km. Note that we could still easily blow up Bradley if we put it out on the quad.

Part 1: Determination of the Drag Coefficient

For this part of the problem we don't need to worry about the Coriolis force. Since the projectile goes so high; however, we assume that the density of the atmosphere, and thereby the drag force, decreases exponentially with height. Since the shell is fairly large and moving very fast, we will neglect the linear part of the drag and only consider the quadratic term. Taking into account these various effects, we write our drag force as

$$F_{drag} = -\kappa e^{-z/L}v^2\hat{\mathbf{v}}$$

where z is up, L is the scale height of the atmosphere (approximately 10 km) and $\hat{\mathbf{v}}$ is a unit vector in the direction of the velocity. Some simple algebra shows that

$$F_{drag} = -\kappa v e^{-z/L}\Big[v_x\hat{\mathbf{e}}_x + v_y\hat{\mathbf{e}}_y + v_z\hat{\mathbf{e}}_x\Big]$$

Breaking this into components, we have

$$\frac{d^2x}{dt^2} = -\beta v v_x \mathrm{e}^{-z/L}$$

$$\frac{d^2y}{dt^2} = -\beta v v_y \mathrm{e}^{-z/L}$$

$$\frac{d^2z}{dt^2} = -g - \beta v v_z \mathrm{e}^{-z/L}$$

where we have defined $\beta = \kappa/m$.

- Task 1: For the given muzzle velocity of 1450 m/s, elevation angle of 55° and atmospheric scale height of 10 km, find the value of β such that the range of the projectile is 120.0 km . (Note: You may assume that all of the motion occurs in the x-z plane since we are neglecting the Coriolis force.)

- Task 2: Once you have found β, use this value and vary the elevation angle from zero to ninety degrees in one degree increments to find the rage as a function of elevation angle. (Make a plot of Range as a function of Elevation Angle) At what angle do you get the maximum range? In class we found that when a uniform density atmosphere caused the drag, we wanted to shoot below 45 degrees to get the maximum range. Do your results for the variable height atmosphere show the same result? If not, why not? (Hint: It may help you to plot the force on the projectile as a function of time for a few elevation angles.)

Part 2: Effects of the Coriolis Force on Targeting

Now lets include the effects of the earth's rotation, i.e. the Coriolis force. Our equation of motion the becomes

$$\mathbf{F} = \mathbf{F}_{gravity} + \mathbf{F}_{drag} + \mathbf{F}_{Coriolis}$$

or

$$m\mathbf{a} = -mg\hat{\mathbf{e}}_z - \kappa \mathrm{e}^{-z/L} v^2 \hat{\mathbf{v}} - 2m(\boldsymbol{\omega} \times \mathbf{v}_r)$$

Once again breaking this into components, we have

$$\frac{d^2x}{dt^2} = -\beta v v_x \mathrm{e}^{-z/L} - 2\omega\Big[v_z \sin\lambda - v_y \cos\lambda\Big]$$

$$\frac{d^2y}{dt^2} = -\beta v v_y \mathrm{e}^{-z/L} - 2\omega v_x \sin\lambda$$

$$\frac{d^2z}{dt^2} = -g - \beta v v_z \mathrm{e}^{-z/L} - 2\omega v_x \cos\lambda$$

where we have used our standard coordinate system, i.e x is east, y is north, z is up and λ is the latitudinal angle.

- Task 3: Using an elevation angle of $55°$ and assuming that we are shooting at a target that is due east of us, by how much will we miss our target in the North-South direction if we neglect the Coriolis force. Plot a graph of x *vs* y to show the deflection as a function of position (Assume $\lambda = 45°$).

- Task 4: At what angles (elevation and azimuthal) should we aim Big Bertha in order to hit a target that is 100 km due east of us? (Note that there should be two answers to this question; you only need to find one.) You may assume a hit if you come within 50 meters of the Target. Make a plot of x *vs* y to show the deflection as a function of position for this case.