1) a) $\underline{F} = F_0 + ct = m\frac{dv}{dt}$

$d\underline{v} = \frac{F_0}{m}dt + \frac{c}{m}t\,dt$

$\int_0^v dv = \frac{F_0}{m}\int_0^t dt + \frac{c}{m}\int_0^t t\,dt$

$$\boxed{\underline{V} = \frac{F_0}{m}t + \frac{c}{2m}t^2}$$

$\frac{dx}{dt} = \frac{F_0}{m}t + \frac{c}{2m}t^2$

$dx = \frac{F_0}{m}t\,dt + \frac{c}{2m}t^2dt$

$\int_0^x dx = \frac{F_0}{m}\int_0^t t\,dt + \frac{c}{2m}\int_0^t t^2dt$

$$\boxed{x = \frac{F_0}{2m}t^2 + \frac{c}{6m}t^3}$$

b) $\underline{F} = F_0\sin(ct) = m\frac{dv}{dt}$

$\int_0^V dV = \frac{F_0}{m}\int_0^t \sin(ct)\,dt$

$V = -\frac{F_0}{cm}\cos(ct)\Big|_0^t$

$$\boxed{V = \frac{F_0}{cm}\left(1-\cos(ct)\right)}$$

$\int_0^x dx = \int_0^t \frac{F_0}{cm}dt - \frac{F_0}{cm}\int_0^t \cos(ct)\,dt$

$x = \frac{F_0}{cm}t - \frac{F_0}{c^2m}\sin(ct)\Big|_0^t$

$$\boxed{x = \frac{F_0}{cm}t - \frac{F_0}{mc^2}\sin(ct)}$$

c) $F = F_0e^{ct} = m\frac{dv}{dt}$

$\int_0^V dV = \frac{F_0}{m}\int_0^t e^{ct}dt$

$= \frac{F_0}{mc}e^{ct}\Big|_0^t$

$$\boxed{\underline{V} = \frac{F_0}{mc}\left[e^{ct}-1\right]}$$

$\int_0^t dx = \frac{F_0}{mc}\int_0^t e^{ct}dt - \frac{F_0}{mc}\int_0^t dt$

$x = \frac{F_0}{mc^2}e^{ct}\Big|_0^t - \frac{F_0}{mc}t$

$$\boxed{x = \frac{F_0}{mc^2}\left(e^{ct}-1\right) - \frac{F_0}{mc}t}$$

2) a) $F = F_0 + cx = mv\frac{dv}{dx}$

$\int_0^V v\,dV = \frac{F_0}{m}\int_0^x dx + \frac{c}{m}\int_0^x x\,dx$

$\frac{1}{2}V^2 = \frac{F_0}{m}x + \frac{c}{2m}x^2$

$$\boxed{V = \left[\frac{2F_0}{m}x + \frac{c}{m}x^2\right]^{1/2}}$$

b) $F = F_0e^{-cx} = mv\frac{dv}{dx}$

$\int_0^V V\,dV = \frac{F_0}{m}\int_0^x e^{-cx}dx$

$\frac{1}{2}V^2 = -\frac{F_0}{mc}e^{-cx}\Big|_0^x$

$V^2 = \frac{2F_0}{mc}\left(1-e^{-cx}\right)$

$$\boxed{V = \sqrt{2F_0/mc}\left(1-e^{-cx}\right)^{1/2}}$$

c) $F = F_0\cos(cx) = mv\,dV/dx$

$\int_0^V V\,dV = \frac{F_0}{m}\int_0^x \cos(cx)\,dx$

$\frac{1}{2}V^2 = \frac{F_0}{mc}\sin(cx)\Big|_0^x = \frac{F_0}{mc}\sin cx$

$$\boxed{V = \sqrt{\frac{2F_0}{mc}}\sqrt{\sin(cx)}}$$

3) On the way up..

$$mv\frac{dv}{dx} = -mg - c_2 v^2$$

Defn $\quad V_t^2 = \frac{mg}{c_2}$

$$mv\frac{dv}{dx} = -c_2\left[\frac{mg}{c_2} + v^2\right]$$

$$v\,dv = -\frac{c_2}{m}\left[V_t^2 + v^2\right]dx$$

$$\int_{v_0}^{0}\frac{v\,dv}{V_t^2+v^2} = -\frac{c_2}{m}\int_0^h dx$$

$$\frac{1}{2}\ln\left(v^2+V_t^2\right)\Big|_{v_0}^{0} = -\frac{c_2}{m}h$$

$$\ln(V_t^2) - \ln\left(v_0^2+V_t^2\right) = -\frac{2c_2}{m}h$$

$$\frac{2c_2}{m}h = \ln\left(\frac{V_0^2+V_t^2}{V_t^2}\right)$$

On the way down....

$$mv\frac{dv}{dx} = -mg + c_2 v^2$$

$$= -c_2\left[\frac{mg}{c_2} - v^2\right]$$

$$\int_0^{v}\frac{v\,dv}{V_t^2-v^2} = -\frac{c_2}{m}\int_h^0 dx$$

$$-\frac{1}{2}\ln\left(V_t^2-v^2\right)\Big|_0^{v} = -\frac{c_2}{m}x\Big|_h^0$$

$$-\frac{1}{2}\ln\left(V_t^2-v^2\right)+\frac{1}{2}\ln\left(V_t^2\right) = \frac{c_2}{m}h$$

$$\frac{2c_2}{m}h = \ln\left(\frac{V_t^2}{V_t^2-v^2}\right)$$

Now $\quad \ln\left(\frac{v_0^2+V_t^2}{V_t}\right) = \frac{2c_2}{m}h = \ln\left(\frac{V_t^2}{V_t^2-v^2}\right)$

Thus

$$\frac{V_0^2+V_t^2}{V_t^2} = \frac{V_t^2}{V_t^2-V^2}$$

$$V_t^2 - V^2 = \frac{V_t^4}{V_0^2+V_t^2}$$

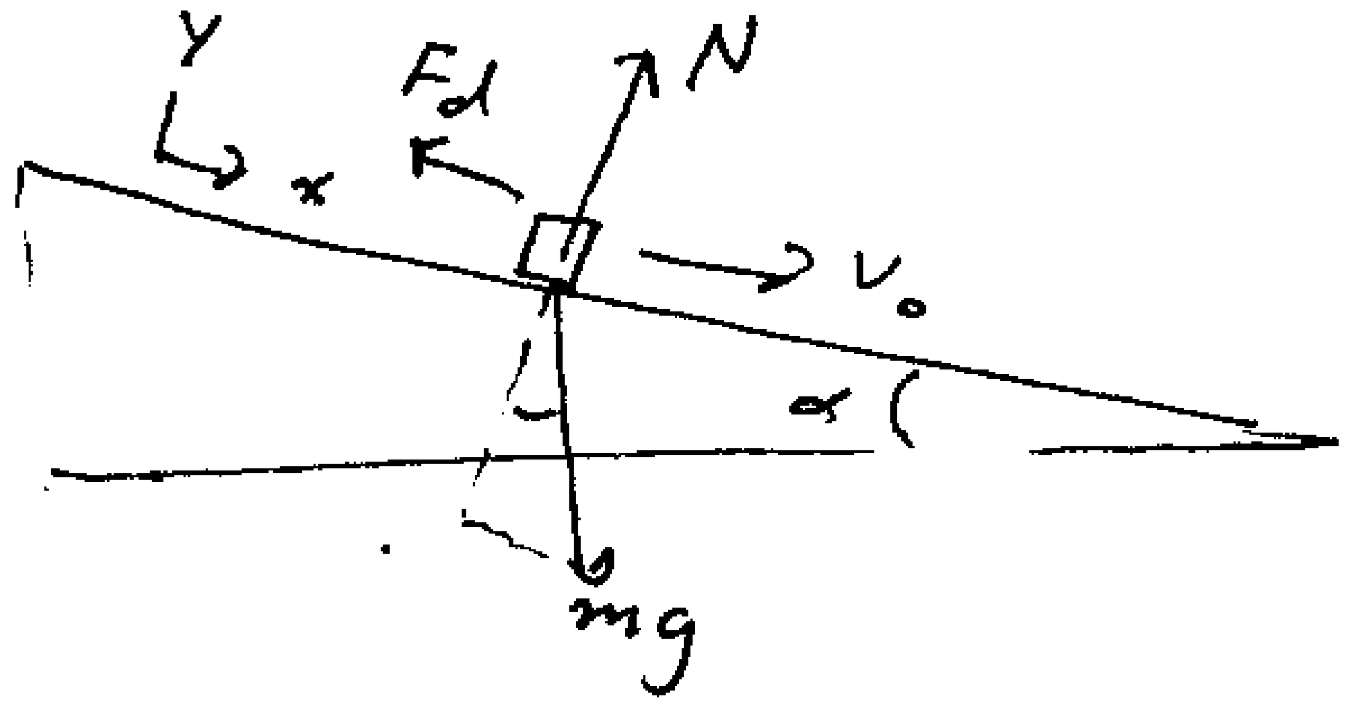
$$V^2 = V_t^2 - \frac{V_t^4}{V_0^2+V_t^2} = \frac{V_t^2\left(V_0^2+V_t^2\right) - V_t^4}{V_0^2+V_t^2}$$

$$V^2 = \frac{V_t^2 V_0^2}{V_0^2+V_t^2}$$

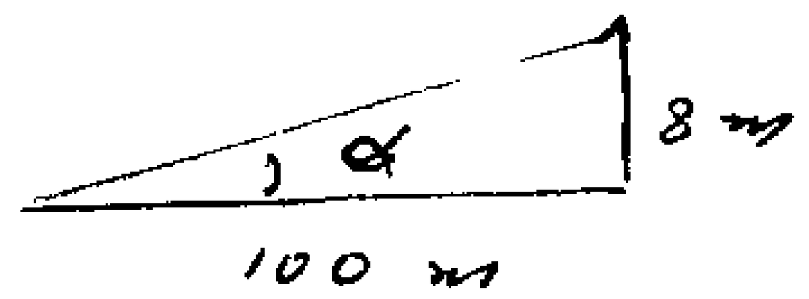
$$\boxed{V = \frac{V_t V_0}{\sqrt{V_0^2+V_t^2}}}$$

QED.

4)



8% grade: Road rises 8 m for every 100 m horizontal dist.



$\tan\alpha = 0.08$

y: dir

$$m\ddot{y} = N - mg\cos\alpha = 0$$

$$\Rightarrow N = mg\cos\alpha$$

x: dir

$$m\ddot{x} = mg\sin\alpha - \mu_k N$$

$$= mg\sin\alpha - \mu_k mg\cos\alpha$$

$$\ddot{x} = g\left[\sin\alpha - \mu_k\cos\alpha\right] = \tilde{g} = \text{const.}$$

$$\ddot{x} = v\frac{dv}{dx} = \tilde{g}$$

$$\int_{v_0}^{0} v\,dv = \tilde{g}\int_0^x dx$$

$$-\tfrac{1}{2}v_0^2 = \tilde{g}x \Rightarrow x = \frac{-v_0^2}{2\tilde{g}} = \frac{-v_0^2}{2g\left[\sin\alpha - \mu_k\cos\alpha\right]}$$

If the car is traveling $25\,\frac{mi}{hr} = 11.1\ m/s$

given $\mu_k = 0.45$

and $\alpha \approx \tan\alpha = 0.08$ rad

$$x = \frac{-(11.1)^2\ m^2/s^2}{2(9.8\ m/s^2)\left[\sin(0.08) - 0.45\cos(0.8)\right]} \approx 17\,m$$

Thus, if the driver had been obeying the speed limit, he could have stopped in 17 m, well before hitting the parked car.

5) a) $m \frac{dV}{dt} = -\alpha e^{\beta V}$

$$\int_{V_0}^{V} e^{-\beta V} dV = -\frac{\alpha}{m}\int_0^t dt$$

$$-\frac{1}{\beta} e^{-\beta V}\Big|_{V_0}^{V} = -\frac{\alpha}{m} t$$

$$e^{-\beta V} - e^{-\beta V_0} = \frac{\alpha\beta}{m} t$$

$$e^{-\beta V} = \frac{\alpha\beta}{m} t + e^{-\beta V_0}$$

$$\boxed{V = -\frac{1}{\beta} \ln\left[\frac{\alpha\beta}{m} t + e^{-\beta V_0}\right]}$$

b) time required to stop

$v = 0$

$$\Rightarrow \ln\left[\frac{\alpha\beta}{m} t + e^{-\beta V_0}\right] = 0$$

$$\frac{\alpha\beta}{m} t + e^{-\beta V_0} = 1$$

$$\boxed{t = \frac{m}{\alpha\beta}\left(1 - e^{-\beta V_0}\right)}$$

c) distance to stop

$$mV \frac{dV}{dx} = -\alpha e^{\beta V}$$

$$\int_{V_0}^{0} V e^{-\beta V} dV = -\frac{\alpha}{m}\int_0^{X_m} dx$$

$$+\frac{1}{\beta} e^{-\beta v}\left(v + \frac{1}{\beta}\right)\Big|_{V_0}^{0} = +\frac{\alpha}{m} X_m$$

$$\boxed{X_m = \frac{m}{\alpha\beta}\left[\frac{1}{\beta} - e^{-\beta V_0}\left(V_0 + \frac{1}{\beta}\right)\right]}$$

6) Given $V(x) = \frac{\alpha}{x}$

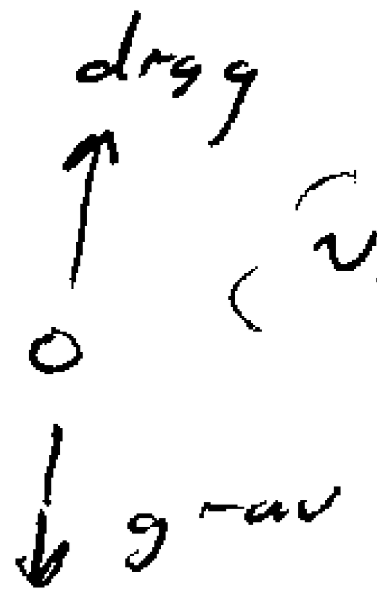
$$mV \frac{dV}{dx} = F(x)$$

$$F(x) = m\left(\frac{\alpha}{x}\right) \frac{d}{dx}\left(\alpha x^{-1}\right)$$

$$= -\frac{m\alpha^2}{x}\left(x^{-2}\right)$$

$$\boxed{F(x) = -\frac{m\alpha^2}{x^3}}$$

7) Balance drag with gravity



(v_t defined as > 0)

$$mg = c_1 v_t + c_2 v_t^2$$

$$c_2 v_t^2 + c_1 v_t - mg = 0$$

$$v_t = \frac{-c_1 \pm \sqrt{c_1^2 + 4 c_2 mg}}{2 c_2}$$

(keep + root to make $v_t > 0$)

$$\boxed{V_t = \sqrt{\left(\frac{c_1}{2c_2}\right)^2 + \left(\frac{mg}{c_2}\right)} - \left(\frac{c_1}{2c_2}\right)}$$

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8) $F = -kx + \dfrac{kx^3}{\alpha^2}$

a) $\dfrac{dU}{dx} = -F = kx - \dfrac{kx^3}{\alpha^2}$

$$U = \frac{1}{2}kx^2 - \frac{1}{4}k\frac{x^4}{\alpha^2} + C$$

$U(0) = 0 \Rightarrow C = 0$

$$\boxed{U(x) = \frac{1}{2}kx^2 - \frac{1}{4}k\frac{x^4}{\alpha^2}}$$

b) Eq. Pt $\Rightarrow F = 0$

$$F = kx\left[1 - \frac{x^2}{\alpha^2}\right] = 0$$

Equil at

$$\boxed{x = 0, \pm\alpha}$$

$$\frac{dU}{dx} = kx - \frac{kx^3}{\alpha^2}$$

$$\frac{d^2U}{dx^2} = k - \frac{3kx^2}{\alpha^2}$$

at $x = 0$ $\quad \dfrac{d^2U}{dx^2} = k > 0$

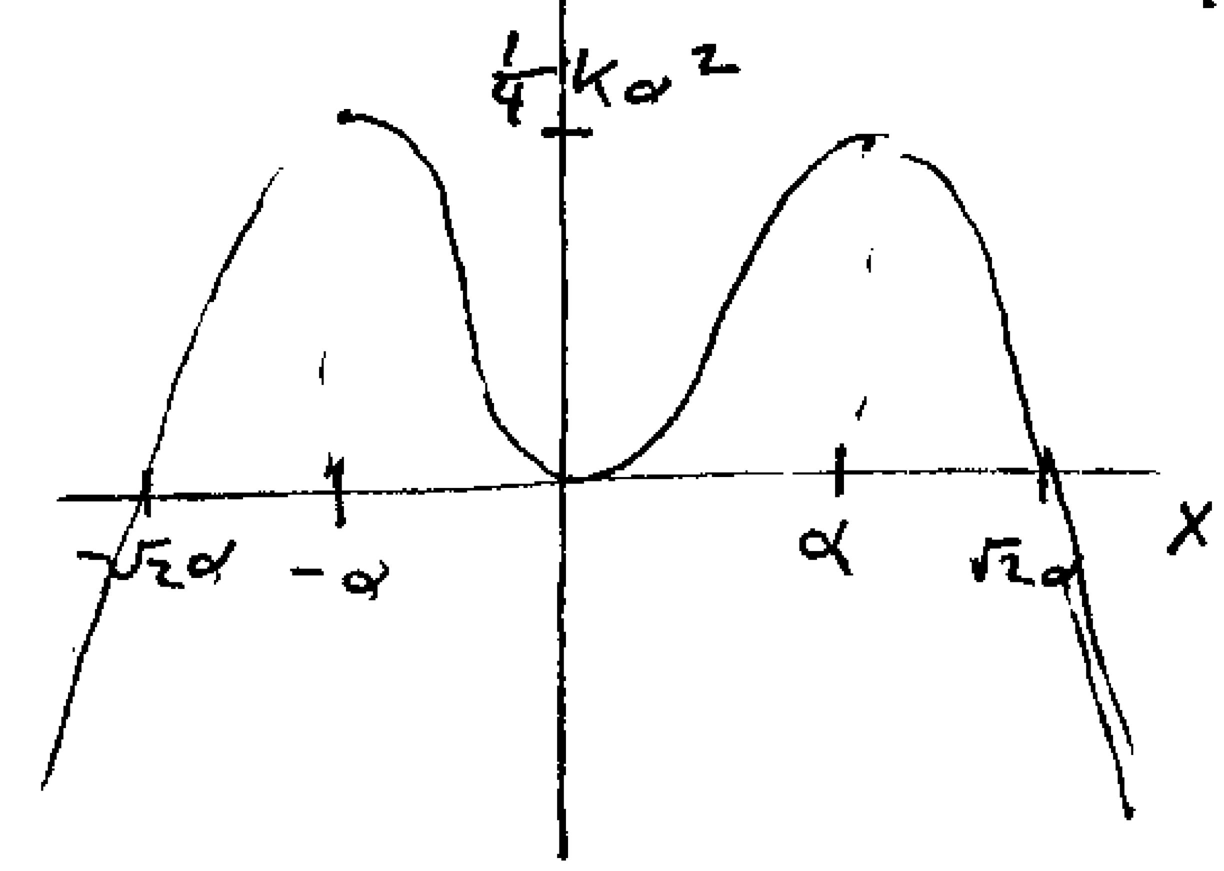
$x = \pm\alpha$ $\quad \dfrac{d^2U}{dx^2} = -2k < 0$

Equil. at 0 stable
$\pm\alpha$ unstable

c)
$$U(\pm\alpha) = \frac{1}{2}k\alpha^2 - \frac{1}{4}k\frac{\alpha^4}{\alpha^2} = \frac{1}{4}k\alpha^2$$

$$U = 0 = \frac{1}{2}kx^2\left[1 - \frac{1}{2}\frac{x^2}{\alpha^2}\right]$$

Roots of U at $x = 0$
$x = \pm\sqrt{2}\,\alpha$



D) Max energy for bound orbit is $\frac{1}{4}k\alpha^2$

E) turning Pt $\left(E < \frac{1}{4}k\alpha^2\right)$

$$E = \frac{1}{2}kx_t^2 - \frac{1}{4}k\frac{x_t^4}{\alpha^2}$$

$$x_t^4 - 2\alpha^2 x_t^2 + \frac{4E\alpha^2}{k} = 0$$

$$x_t^2 = 2\alpha^2 \pm \sqrt{4\alpha^2 - \frac{16E\alpha^2}{k}}$$

$$x_t = \pm\sqrt{2\alpha^2 \pm \sqrt{4\alpha^2 - \frac{16E\alpha^2}{k}}}$$