# Homework #10

1) (M&T 11.22) Show that the trace of the inertia tensor is invariant under a similarity transformation.
2) (M&T 11.27) A symmetric body moves without the influences of forces or torques. Let $x_3$ be the symmetry axis of the body and **L** be along $x_3'$. The angle between $\boldsymbol{\omega}$ and $x_3$ is $\alpha$. Let $\boldsymbol{\omega}$ and **L** be initially in the $x_2 - x_3$ plane. What is the angular velocity of the symmetry axis about **L** in terms of $I_1$, $I_3$, $\omega$ and $\alpha$?
3) (M&T 11.29) Investigate the motion of the symmetric top for the case in which the axis of rotation is vertical ($x_3 = x_3'$) Shown that the motion is either stable or unstable depending on whether the quantity $4I_1Mgh/I_3^2\omega_3^2$ is less than or greater than unity.

   Sketch the effective potential for the two cases and point out the features of the curves that determine the stability. If the top is set spinning in the stable configuration, what is the effect of friction gradually reducing the value of $\omega_3$?

1) we know that

$$I'_{ij} = \sum_{k,l} \lambda_{ik} I_{kl} \lambda^{-1}_{lj}$$

By Def$^{n}$,

$$Tr\{\underline{\underline{I}}'\} = \sum_{i} I'_{ii}$$

(i.e. the sum of the Diagonal elements

$$= \sum_{i} \sum_{k,l} \lambda_{ik} I_{kl} \lambda^{-1}_{li}$$

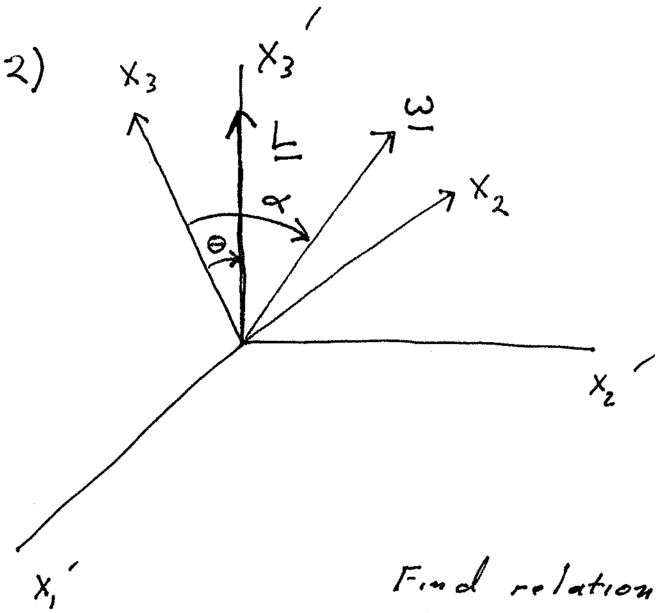
$$= \sum_{k,l} I_{kl} \sum_{i} \lambda_{ik} \lambda^{-1}_{li}$$

$$= \sum_{k,l} I_{kl} \delta_{kl}$$

$$= \sum_{l} I_{ll}$$

$$= Tr\{\underline{\underline{I}}\}$$ Q.E.D.

2)




$I_1 = I_2 \neq I_3$

$\underline{\omega}$ + $\underline{L}$ initially in $x_2 - x_3$ Plane

Initially

$$L_1 = I_1 \omega_1 = 0$$

$$L_2 = I_1 \omega_2 = L \sin\theta = I_1 \omega \sin\alpha$$

$$L_3 = I_3 \omega_3 = L \cos\theta = I_3 \omega \cos\alpha$$

Find relation between $\theta$ + $\alpha$

$$\tan\theta = \frac{I_1}{I_3} \tan\alpha$$

~~Kgasm~~ By Defn

$$\omega_3 = \dot\phi \cos\theta + \dot\psi$$

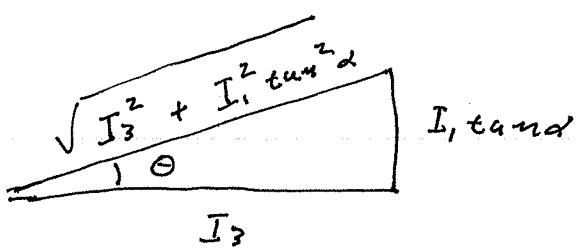
OR

$$\dot\phi \cos\theta = \omega_3 - \dot\psi$$

But $$\dot\psi = -\Omega = -\frac{I_3 - I_1}{I_1} \omega_3 = \omega_3 - \frac{I_3}{I_1} \omega_3$$

$$\dot\phi \cos\theta = \omega_3 + \frac{I_3}{I_1} \omega_3 - \omega_3 = \frac{I_3}{I_1} \omega_3$$

$$\dot\phi \cos\theta = \frac{I_3}{I_1} \omega \cos\alpha$$




$$\cos\theta = \frac{I_3}{\sqrt{I_3^2 + I_1^2 \tan^2\alpha}}$$

OR

$$\dot\phi = \frac{\omega_0}{I_1} \cos\alpha \sqrt{I_3^2 + I_1^2 \frac{\sin^2\alpha}{\cos^2\alpha}} = \frac{\omega}{I_1} \sqrt{I_3^2 \cos^2\alpha + I_1^2 \sin^2\alpha}$$

3) When the motion is vertical $(\theta = 0)$ $(\dot{\theta} = 0)$

$$\left.\begin{array}{l} P_\phi = I_3 \dot{\phi} + I_3 \dot{\psi} \\ P_\psi = I_3 \dot{\phi} + I_3 \dot{\psi} \end{array}\right\} \Rightarrow P_\phi = P_\psi = I_3 \omega_3$$

The energy is

$$E = \tfrac{1}{2} I_3 \omega_3^2 + mgh$$

$$E' = \left(E - \tfrac{1}{2} I_3 \omega_3^2\right) = mgh$$

To examine the stability of the system, consider the general Energy equation.

$$E' = \tfrac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + mgh\cos\theta$$

$$= \frac{1}{2} I_1 \dot{\theta}^2 + \frac{(P_\phi - P_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + mgh\cos\theta = mgh$$

consider small angles

$$E' = \tfrac{1}{2} I_1 \dot{\theta}^2 + \frac{P_\psi^2 \left(1 - \left[1 - \frac{\theta^2}{2}\right]\right)^2}{2 I_1 (\theta^2)} + mgh\left(1 - \frac{\theta^2}{2}\right) \cancel{= mgh}$$

$\Rightarrow$

$$E' = \tfrac{1}{2} I_1 \dot{\theta}^2 + \frac{I_3^2 \omega_3^2 \left[\theta^4/4\right]}{2 I_1 \theta^2} + mgh\left(1 - \frac{\theta^2}{2}\right) \cancel{- mgh = 0}$$

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{I_3^2 \omega_3^2}{8 I_1} \theta^2 - \tfrac{1}{2} mgh\,\theta^2 - mgh$$

$$E'' = E' - mgh = \tfrac{1}{2} I_1 \dot{\theta}^2 + \tfrac{1}{2}\left[\frac{I_3^2 \omega_3^2}{4 I_1} - mgh\right]\theta^2$$

Note: This is of the form

$$E = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} k \theta^2$$

$$k = \frac{I_3^2 \omega_3^2}{4 I_1} - mgh$$

For oscillations $k > 0$

$$\frac{I_3^2 \omega_3^2}{4 I_1} - mgh > 0$$

$$\frac{I_3^2 \omega_3^2}{4 I_1} > mgh$$

$$1 > \frac{4 mgh I_1}{I_3^2 \omega_3^2} \qquad \text{QED}$$

Now

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + \underbrace{\frac{I_3^2 \omega_3^2}{2 I_1} \left[ \frac{(1 - \cos\theta)^2}{\sin^2\theta} \right] + mgh\cos\theta}_{V(\theta)}$$

$$V(\theta) = \frac{I_3^2 \omega_3^2}{2 I_1} \frac{(1 - \cos\theta)^2}{\sin^2\theta} + mgh\cos\theta$$

$$= \frac{I_3^2 \omega_3^2}{4 I_1} \left[ \frac{2(1 - \cos\theta)^2}{\sin^2\theta} + \frac{4 I_1 mgh}{I_3^2 \omega_3^2} \cos\theta \right]$$

Let $\beta = \dfrac{4 I_1 m g h}{I_3^2 \omega_3^2}$

$$V(\theta) = \frac{I_3^2 \omega_3^2}{4 I_1} \left\{ \frac{2(1-\cos\theta)^2}{\sin^2\theta} + \beta \cos\theta \right\}$$

$$= \frac{I_3^2 \omega_3^2}{4 I_1} \underbrace{\left\{ \frac{2(1-\cos\theta)}{1+\cos\theta} + \beta \cos\theta \right\}}_{f(\theta)}$$

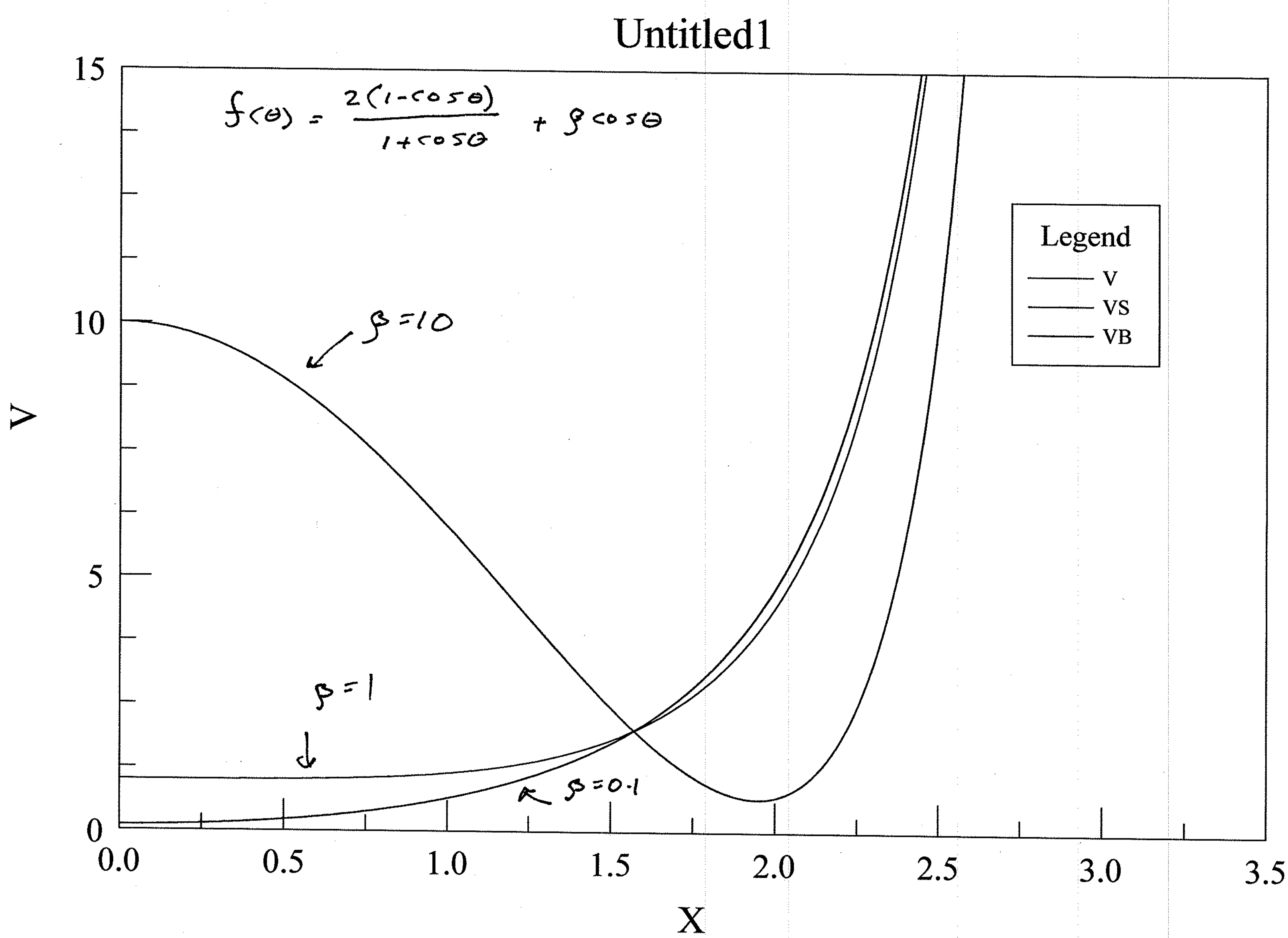
see graph.

~~Note that we only have a local min if $\beta > 1$ is the stability condition.~~

Note that for small $\theta$, if $\beta > 1$ a small perturbation ~~exists, then~~ grows

 local max

But for $\beta < 1$ a small pert. is returned to equil. (local min).


Untitled1
$f(\theta) = \frac{2(1-\cos\theta)}{1+\cos\theta} + \beta\cos\theta$
$\beta = 10$
$\beta = 1$
$\beta = 0.1$
Legend
V
VS
VB
15
10
5
0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
V
X