Homework #9

1) (M&T 11.10)A solid sphere of mass M and radius R rotates freely in space with an angular velocity ω about a fixed diameter. A particle of mass m, initially at one pole, moves with a constant speed v along a great circle of the sphere. Show that when the particle has reached the other pole, the rotation of the sphere will have been retarded by an angle

$$\alpha = \omega T\left(1 - \sqrt{\frac{2M}{2M+5m}}\right)$$

where T is the total time required for the particle to move from one pole to the other.

2) (M&T 11.13) A three particle system consists of mass m_i and coordinates (x_1, x_2, x_3) as follows:

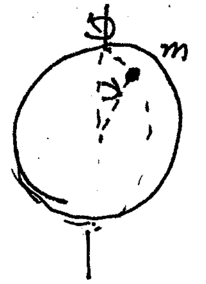
$$m_1 = 3m, \quad (b, 0, b)$$
$$m_2 = 4m, \quad (b, b, -b)$$
$$m_3 = 2m, \quad (-b, b, 0)$$

Find the inertia tensor, the principal moments of inertia and the principal axes.

3) Consider a thin homogeneous plate of arbitrary shape that lies in the x_1-x_2 plane. Show that the inertia tensor takes on the form

$$\{I\} = \begin{Bmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{Bmatrix}$$

(M+T 11.10)



$I_{sp} = \frac{2}{5} MR^2$

when the pt. mass is at the pole, the moment of inertia of the system is just the ~~I~~ of the sphere.

when the mass has moved to an angle Θ, the moment of the system is

$$I = \frac{2}{5} MR^2 + m\,(R\sin\Theta)^2$$

since the are no external forces present

$$L_0 = L_z = \left(\frac{2}{5} MR^2\right)\omega$$

as the mass moves we have

$$L_z = \left(\frac{2}{5} MR^2 + mR^2\sin^2\Theta\right)\dot{\phi}$$

$$\Rightarrow \quad \dot{\phi} = \frac{\left(\frac{2}{5} MR^2\right)\omega}{\left(\frac{2}{5} MR^2 + mR^2\sin^2\Theta\right)}$$

But $\Theta = vt/R$

$$\dot{\phi} = \frac{\frac{2}{5} MR^2\omega}{\frac{2}{5} MR^2 + mR^2\sin^2\left(\frac{vt}{R}\right)}$$

$$\dot{\phi} = \frac{\omega}{1 + \frac{5}{2}\frac{m}{M}\sin^2\left(\frac{vt}{R}\right)}$$

$$\phi = \int_{t=0}^{t=\frac{\pi R}{V}} \frac{\omega\, dt}{1 + \frac{5}{2}\frac{m}{M}\sin^2\left(\frac{Vt}{R}\right)}$$

let $u = \frac{Vt}{R} \Rightarrow du = \frac{V}{R}dt$; $dt = \frac{R}{V}du$

whe $t = 0$, $u = 0$

$t = \frac{\pi R}{V} \Rightarrow u = \frac{V}{R}\frac{\pi R}{V} = \pi$

$$\phi = \frac{\omega R}{V}\int_0^{\pi} \frac{du}{1 + \beta \sin^2(u)} \qquad \beta = \frac{5m}{2M}$$

$$= \frac{2\omega R}{V}\int_0^{\pi/2} \frac{du}{1 + \beta \sin^2 u}$$

But $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$

$$1 + \beta \sin^2 u = 1 + \frac{\beta}{2}(1 - \cos(2u))$$

$$= (1 + \beta/2) - \frac{\beta}{2}\cos(2u)$$

$$\phi = \frac{2\omega R}{V}\int_0^{\pi/2} \frac{du}{(1 + \beta/2) - \beta/2 \cos(2u)}$$

Let $x = 2u$ when $u = 0$, $x = 0$

$dx = 2du$ $u = \pi/2$, $x = \pi$

$$\phi = \frac{\omega R}{V}\int_0^{\pi} \frac{dx}{(1+\beta/2) - \beta/2 \cos x}$$

But

$$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}}\, \mathrm{Tan}^{-1}\left[\frac{(a-b)\tan\left(\frac{x}{2}\right)}{\sqrt{a^2-b^2}}\right]$$

$$a = 1+\beta/2 \qquad b = -\beta/2$$

$$a^2-b^2 = (1+\beta/2)^2 - (\beta/2)^2$$

$$= 1+\beta+\beta^2/4 - \beta^2/4 = 1+\beta$$

$$a-b = 1+\beta/2 - (-\beta/2) = 1+\beta$$

$$= \frac{2}{\sqrt{1+\beta}}\, \mathrm{Tan}^{-1}\left[\frac{(1+\beta)\tan\left(\frac{x}{2}\right)}{\sqrt{1+\beta}}\right]$$

$$= \frac{2}{\sqrt{1+\beta}}\, \mathrm{Tan}^{-1}\left[\sqrt{1+\beta}\,\tan\left(\frac{x}{2}\right)\right]$$

$$\phi = \frac{\omega R}{V}\left(\frac{2}{\sqrt{1+\beta}}\right)\left(\mathrm{Tan}^{-1}\left[\sqrt{1+\beta}\,\tan\left(\frac{\pi}{2}\right)\right] - \mathrm{Tan}^{-1}\left[\sqrt{1+\beta}\,\tan(0)\right]\right)$$

$$= \frac{\omega R}{V}\left(\frac{2}{\sqrt{1+\beta}}\right)\left\{\underbrace{\mathrm{Tan}^{-1}(\infty)}_{\pi/2} - \cancel{\mathrm{Tan}^{-1}(0)}\right\}$$

$$= \frac{\pi\omega R}{\cancel{2}V}\,\frac{\cancel{2}}{\sqrt{1+\beta}} = \frac{\pi\omega R}{V}\,\frac{1}{\sqrt{1+5m/2M}}$$

• But $\pi R = VT$

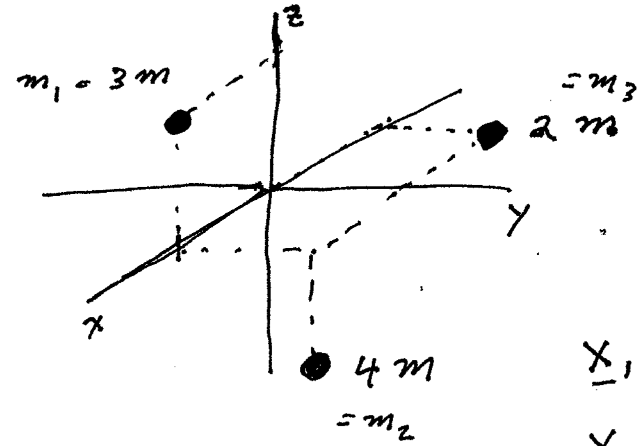
• $$\phi = \omega T \sqrt{\frac{2M}{2M+5m}}$$

If we set $m = 0$ (or just not have mass make transit, the sphere would have rotated through an angle $\phi_0 = \omega T$ in a time T. Thus the sphere/mass system is retarded by an amount

$$\Delta\phi = \phi_0 - \phi$$

$$\boxed{\Delta\phi = \omega T \left[1 - \sqrt{\frac{2M}{2M+5m}} \right]}$$

M+T 11-13



$$I_{ij} = \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k x_{\alpha k}^2 - x_{\alpha i} x_{\alpha j} \right]$$

$$\underline{x}_1 = (b, 0, b)$$
$$\underline{x}_2 = (b, b, -b)$$
$$\underline{x}_3 = (-b, b, 0)$$

$$I_{11} = m_1 (x_{12}^2 + x_{13}^2) + m_2 (x_{22}^2 + x_{23}^2) + m_3 (x_{32}^2 + x_{33}^2)$$
$$= 3m [b^2] + 4m [b^2 + b^2] + 2m [b^2] = 13 m b^2$$

$$I_{22} = m_1 (x_{11}^2 + x_{13}^2) + m_2 (x_{21}^2 + x_{23}^2) + m_3 (x_{31}^2 + x_{33}^2)$$
$$= 3m [b^2 + b^2] + 4m [b^2 + b^2] + 2m [b^2] = 16 m b^2$$

$$I_{33} = m_1 (x_{11}^2 + x_{12}^2) + m_2 (x_{21}^2 + x_{22}^2) + m_3 (x_{31}^2 + x_{32}^2)$$
$$= 3m [b^2] + 4m [b^2 + b^2] + 2m [b^2 + b^2] = 15 m b^2$$

$$I_{12} = I_{21} = -m_1 x_{11} x_{12} - m_2 x_{21} x_{22} - m_3 x_{31} x_{32}$$
$$= -3m [\cancel{b \cdot 0}] - 4m [b \cdot b] - 2m [-b \cdot b]$$
$$= -4 m b^2 + 2 m b^2 = -2 m b^2$$

$$I_{13} = I_{31} = -m_1 x_{11} x_{13} - m_2 (x_{21} x_{23}) - m_3 (x_{31} x_{33})$$

$$= -3m(b \cdot b) - 4m(b \cdot -b) - 2m(\cancel{-b} \cdot 0)$$

$$= -3mb^2 + 4mb^2 = +1mb^2$$

$$I_{23} = I_{32} = -m_1 x_{12} x_{13} - m_2 (x_{22} x_{23}) - m_3 (x_{32} x_{33})$$

$$= -3m(\cancel{0 \cdot b}) - 4m(b \cdot -b) - 2m(\cancel{b \cdot 0})$$

$$= +4mb^2$$

$$\underline{\underline{I}} = mb^2 \begin{pmatrix} 13 & -2 & 1 \\ -2 & 16 & 4 \\ 1 & 4 & 15 \end{pmatrix}$$

Find P.M.I.

$$\left| mb^2 \begin{pmatrix} 13-\lambda & -2 & 1 \\ -2 & 16-\lambda & 4 \\ 1 & 4 & 15-\lambda \end{pmatrix} \right| = 0$$

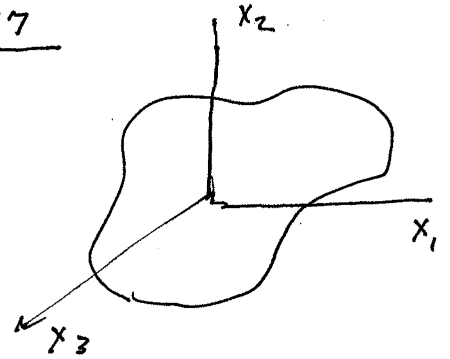
Sheet: Untitled1

	Column1	Column2	Column3	EigValReal4	EigValImg5	EigVec6	EigVec7	EigVec8
1	13.0000	-2.0000	1.0000	10.0000	0.0000	0.5774	0.8052	-0.1355
2	-2.0000	16.0000	4.0000	14.3542	0.0000	0.5774	-0.2852	0.7651
3	1.0000	4.0000	15.0000	19.6458	0.0000	-0.5774	0.5199	0.6295

Eigen values
Note: All 3 are Real!

PA_1 PA_2 PA_3

11-17



assume negligible thickness in x_3-dir

By Def^{n} the elements of the inertia tensor are

$$r^2 = x_1^2 + x_2^2$$

$$I_{11} = \rho_s \int (r^2 - x_1^2)\, dx_1\, dx_2$$

$$= \rho_s \int x_2^2\, dx_1\, dx_2 = A$$

$$I_{22} = \rho_s \int (r^2 - x_2^2)\, dx_1\, dx_2$$

$$= \rho_s \int x_1^2\, dx_1\, dx_2 = B$$

$$I_{33} = \rho_s \int (r^2 - \cancel{x_3^2})\, dx_1\, dx_2$$

$$= \rho_s \int (x_1^2 + x_2^2)\, dx_1\, dx_2 = A + B$$

$$I_{12} = I_{21} = \rho_s \int (-x_1 x_2)\, dx_1\, dx_2 = -C$$

$$I_{13} = I_{31} = \rho_s \int -x_1 \cancel{x_3}\, dx_1\, dx_2 = 0$$

$$I_{23} = I_{32} = \rho_s \int -x_2 \cancel{x_3}\, dx_1\, dx_2 = 0$$

$$\underline{\underline{I}} = \begin{pmatrix} A & -C & 0 \\ -C & B & 0 \\ 0 & 0 & A+B \end{pmatrix}$$