

**Physics 320**  
**Exam #1, Spring 2002**

Name: Key

Three questions, choose any 2. (50 points each.)

1. A particle of mass  $m$  moves in the one dimensional potential given by

$$U(x) = \frac{\kappa}{2} (1 - \exp(-x^2))$$

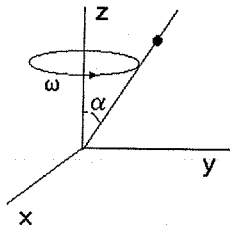
where  $\kappa$  is a positive constant.

- a) What is the force acting on the particle as a function of  $x$ ? Is this a hard or a soft spring? Why? (7 points)
  - b) Sketch the potential and the phase space diagram. Be sure to label important points, e.g. equilibrium, separatrix, etc. (7 points)
  - c) Write the Lagrangian for the particle. (7 points)
  - d) Based on the functional form of the Lagrangian, are there any conserved quantities for this system? If yes, what and why? (7 points)
  - e) Find the equation of motion using the Lagrange equation. (8 points)
  - f) What is the maximum velocity the mass may have at  $x=0$  if the motion is to remain bounded? (7 points)
  - g) Show that for small amplitudes ( $x \ll 1$ ), the mass behaves like a mass a simple harmonic oscillator. (7 points)
2. A bead of mass  $m$  is free to slide along a straight rod that makes an angle  $\alpha$  with respect to the  $z$ -axis. The rod in turn rotates around the  $z$ -axis with a constant angular velocity  $\omega$ . Gravity acts in the minus  $z$  direction. Take  $z=0$  to be the zero potential point.
- a) Using cylindrical coordinates, write the Lagrangian for the system. Explicitly eliminate the  $z$  from the Lagrangian in favor of  $r$  by using the equation of constraint,  $z=r \cot\alpha$ . (You may write that  $\beta=\cot\alpha$ .) (10 points)
  - b) Based on the structure of the Lagrangian, are there any conserved quantities? Why? (10 points)
  - c) Write the Lagrange equations for the motion. (10 points)
  - d) Find the Hamiltonian for the system (10 points)
  - e) Find the Canonical equations of motion (10 points)
3. It is found that in a certain region of space, the index of refraction of the medium is non-uniform and varies as

$$n(x) = n_0 e^{\alpha x}$$

where  $n_0$  and  $\alpha$  are positive constants. Make use of Fermat's Principle (i.e. the actual path taken by a light ray will be the path which take the minimum time) to calculate the path of a light ray traveling from  $x_1, y_1$  to  $x_2, y_2$ . Recall that the speed of light in a medium is given by  $v=c/n$ .

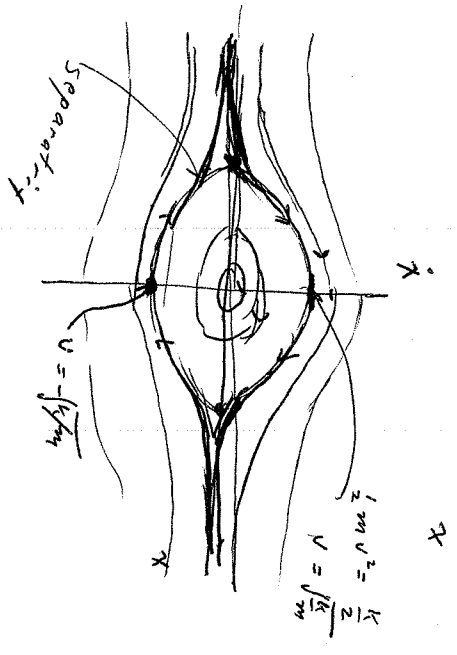
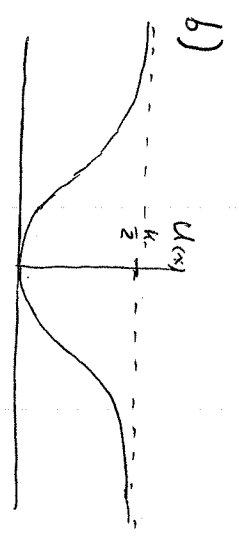
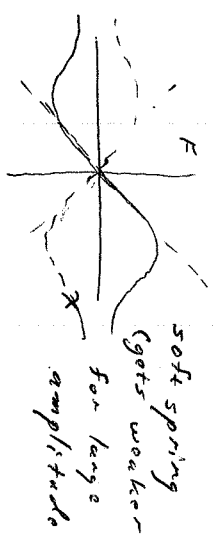
- a) Write the integral  $J$  which is to be minimized. (Hint the problem is easiest if we take the independent variable to be  $x$ .) (10 points)
- b) Use the first form of Euler's equation to find the differential equation for the path  $y(x)$ . (Note that if you used the hint in part a), then  $\partial f/\partial y=0$ .) (15 points)
- c) Let  $\alpha=0$  in your equation and integrate to find the path of the light ray. (10 points)
- d) Now keep  $\alpha \neq 0$  and integrate your expression to find  $y(x)$ . (I found it useful to make the substitution  $u = e^{-\alpha x}$  in performing the integration.) (15 points)



1)  $U(x) = \frac{k}{2} (1 - e^{-x^2})$

a)  $F = -\frac{dU}{dx} = +\frac{k}{2} - 2x e^{-x^2}$

$F = -kx e^{-x^2}$



c)  $\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{k}{2} (1 - e^{-x^2})$

d) Since  $\mathcal{L}$  does not depend on  $t$ .  $\Rightarrow \mathcal{H} = T + U = \text{const}$

e)  $\frac{\partial \mathcal{L}}{\partial x} = \frac{k}{2} \frac{\partial}{\partial x} e^{-x^2}$

$= -kx e^{-x^2}$

$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m \ddot{x}$

$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = m \ddot{x}$

$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = 0$

$m \ddot{x} + kx e^{-x^2} = 0$

f) To remain bound, use conservation of energy

$T(x=0) < U(\infty)$

$\frac{1}{2} m \dot{x}_0^2 = \frac{k}{2}$

$\dot{x}_0 = \sqrt{\frac{k}{m}}$

g) If  $x \ll 1$

$e^{-x^2} \approx 1 - x^2 + \dots$

$m \ddot{x} + kx (1 - x^2) = 0$

$\ddot{x} + \frac{k}{m} x = 0$

SHO equation

a)  $\mathcal{L} = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2) - m g z$

$z = r \cos \alpha = \rho r$

$\dot{z} = \rho \dot{r}$

$\mathcal{L} = \frac{1}{2} m [(1+\beta^2)\dot{r}^2 + r^2 \dot{\theta}^2] - m g \rho r$

b)  $\mathcal{H} \neq T + U$  is conserved since

$\mathcal{L}$  is not an explicit  $\mathcal{H}$

~~$\left( \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = m r^2 \dot{\theta}$  is conserved since  $\theta$  is cyclic~~

~~c)  $\theta$ -equation  $\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \dot{\theta} = \text{const}$~~

~~$\frac{\partial \mathcal{L}}{\partial r} = m r \dot{\theta}^2 - m g \beta = m \omega^2 r - m g \beta$~~

$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m (1+\beta^2) \dot{r} = P_r$

$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{r}} \right) = m (1+\beta^2) \ddot{r}$

$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{r}} \right) - \frac{\partial \mathcal{L}}{\partial r} = 0$

$m (1+\beta^2) \ddot{r} - m \omega^2 r + m g \beta = 0$

d)  $\mathcal{H} = T + U =$

$$= \frac{1}{2} m (1+\beta^2) \dot{r}^2 + \frac{1}{2} m \omega^2 r^2 + mg\beta r$$

But  $\dot{r} = \frac{p_r}{m(1+\beta^2)}$

$$\mathcal{H} = \frac{1}{2} m \frac{p_r^2}{m^2(1+\beta^2)} + \frac{1}{2} m \omega^2 r^2 + mg\beta r$$

$$\mathcal{H} = \frac{p_r^2}{2m(1+\beta^2)} + \frac{1}{2} m \omega^2 r^2 + mg\beta r$$

e)  $\frac{\partial \mathcal{H}}{\partial p_r} = -\dot{p}_r = m \omega^2 r + mg\beta$

$$\dot{p}_r = -m \omega^2 r - mg\beta$$

$$\frac{\partial \mathcal{H}}{\partial p_r} = \dot{r} = \frac{p_r}{m(1+\beta^2)}$$

3)

a)  $n(x) = n_0 e^{\alpha x}$

$$t = \int_{-1}^2 dt = \int_{-1}^2 \frac{ds}{v} = \int_{-1}^2 \frac{ds}{c}$$

$$t = \frac{n_0}{c} \int_{-1}^2 e^{\alpha x} \sqrt{dx^2 + dy^2}$$

$$= \frac{n_0}{c} \int_{-1}^2 e^{\alpha x} \sqrt{1 + \frac{dy}{dx}} dx$$

$$t = \frac{n_0}{c} \int_{x_1}^{x_2} (e^{\alpha x} \sqrt{1+y'^2}) dx$$

b) obviously,  $f = e^{\alpha x} \sqrt{1+y'^2}$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = 0$$

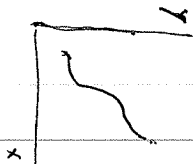
OR  $\frac{\partial f}{\partial y'} = \text{const.} = A$

$$\frac{\partial f}{\partial y'} = e^{\alpha x} \frac{1}{2} \cdot \frac{1}{\sqrt{1+y'^2}} \cdot 2y' = \frac{y' e^{\alpha x}}{\sqrt{1+y'^2}} = A$$

$$y'^2 e^{2\alpha x} = A^2 + A^2 y'^2$$

$$(e^{2\alpha x} - A^2) y'^2 = A^2$$

$$y'^2 = \frac{A^2}{(e^{2\alpha x} - A^2)}$$



$$\frac{dy}{dx} = \frac{A}{\sqrt{e^{2\alpha x} - A^2}}$$

c) I 5  $\alpha = 0$

$$\frac{dy}{dx} = \left( \frac{A}{\sqrt{1-A^2}} \right) = B = \text{const.}$$

$$dy = B dx$$

$$\boxed{y = Bx + C}$$

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d)

$$dy = \frac{A dx}{\sqrt{e^{2\alpha x} - A^2}}$$

$$= \frac{A dx}{e^{\alpha x} \sqrt{1 - A^2 e^{-2\alpha x}}}$$

$$dy = \frac{A e^{-\alpha x} dx}{\sqrt{1 - A^2 e^{-2\alpha x}}}$$

$$y = \int \frac{A e^{-\alpha x} dx}{\sqrt{1 - (A e^{-\alpha x})^2}}$$

let  $u = A e^{-\alpha x}$

$$du = -\alpha A e^{-\alpha x} dx$$

$$\Rightarrow -\frac{1}{\alpha} du = A e^{-\alpha x} dx$$

$$y = -\frac{1}{\alpha} \int \frac{du}{\sqrt{1-u^2}}$$

$$= -\frac{1}{\alpha} \sin^{-1}(u) + C$$

$$\boxed{y = -\frac{1}{\alpha} \sin^{-1} [A e^{-\alpha x}] + C}$$